Pset #3 Solutions

Problem 1.

Because B is continuous the event $\{\limsup_{t\to\infty} B(t) = \infty\}$ is the same as the event $\{\sup_{t\in\mathbb{R}^+} B(t) = \infty\}$ (check from the definition that they are indeed the same. Another definition of $\limsup_{t\to\infty} B(t) = \infty$ is that for all $M > 0$ and all $t > 0$, there exists $t' > t$ such that $B(t') \geq M$). For any $k \geq 0$, A_k be the event $\{\sup_{t\geq 0} B(t) \geq k\}$, and $A = \{\limsup_{t\to\infty} B(t) = \infty\}$. Then, $A = \cap_{1\leq k\leq\infty} A_k$, and thus $\mathbb{P}(A) = \lim_k \mathbb{P}(\cap_{1\leq j\leq k} A_j) = \lim_k \mathbb{P}(A_k)$. Define the following sequence of stopping times for our Brownian motion:

$$t_1 = 1$$
$$t_2 = t_1 + (k + |B(t_1)|)^2$$
$$t_3 = t_2 + (k + |B(t_2)|)^2$$
$$t_{n+1} = t_n + (k + |B(t_n)|)^2$$

$$\mathbb{P}((A_k)^c) \leq \mathbb{P}(\forall t_i,\ B(t_i) \leq k) = \prod_{n\geq 1} \mathbb{P}(B(t_{n+1}) \leq k \mid \forall m \leq n,\ B(t_m) \leq k) = \prod_n \mathbb{P}(B(t_{n+1}) \leq k \mid B(t_n) \leq k)$$

For any n, $\mathbb{P}(B_{t_{n+1}} \leq k \mid B(t_n) \leq k) = \mathbb{P}(B(t_{n+1}) - B(t_n) \leq k - B(t_n) | B_{t_n} \leq k) \leq \mathbb{P}(B(t_{n+1}) - B(t_n) \leq k + |B(t_n)||B_{t_n} \leq k)$. By the strong Markov property of Brownian motion, $B(t_{n+1}) - B(t_n)$ is a Gaussian variable with standard deviation $k + |B(t_n)|$, and hence $\mathbb{P}(B(t_{n+1}) - B(t_n) \leq k + |B(t_n)||B_{t_n} \leq k) \leq \alpha$, where α is the probability $P(X \leq 1)$, where X is a Gaussian variable with variance 1 ($\alpha \sim 0.841$). As a result,

$$\mathbb{P}((A_k)^c) \leq \lim_n \alpha^n = 0$$

and thus $\mathbb{P}(A_k) = 1$, and $\mathbb{P}(A) = 1$ QED.

Problem 2(a)

We use the decomposition

$$Q(\Pi_n, B) - T = \sum_i \left(\left(B(\frac{i+1}{n}) - B(\frac{i}{n}) \right)^2 - (\frac{i+1}{n} - \frac{i}{n}) \right)$$

Each $B(\frac{i+1}{n}) - B(\frac{i}{n})$ is an independent (for n fixed) random variable with variance $\frac{1}{n}$. In other words, we can rewrite

$$\left(B(\frac{i+1}{n}) - B(\frac{i}{n}) \right)^2 = \frac{1}{n} \left(\sqrt{n}B(\frac{i+1}{n}) - \sqrt{n}B(\frac{i}{n}) \right)^2 = \frac{1}{n} X_{i,n}$$

where $X_{i,n}$ is defined as $\left(\sqrt{n}B(\frac{i+1}{n}) - \sqrt{n}B(\frac{i}{n})\right)^2$. Note that the $X_{i,n}$ are all identically distributed random variables (in particular the distribution does not depend on n). Also, $X_{i,n}$ follows a chi-squared distribution (it is the square of a normal), has mean 1 (its mean is the variance of our normal random variable, which was scaled to have variance 1), has bounded moments, in particular bounded second and fourth moments. We obtain that

$$Q(\Pi_n, B) - T = \frac{1}{n} \sum_{i\leq n} (X_{i,n} - 1)$$

We apply our special form of the SLLN and conclude that

$$Q(\Pi_n, B)$$

converges to zero almost surely.

2 Problem 2b (Based on Tetsuya Kaji's solutions)

Proof. For a fixed n, let $A_{n,i} = (B(t_i) - B(t_{i-1}))^2 - (t_i - t_{i-1})$, $t_0 = 0$ and $t_{n+1} = T$. Then $A_{n,i}$ are independent w.r.t. the probability space of Brownian motion. with mean 0 and $Q(\Pi_n, B) - T = \sum_i A_{n,i}$. Consider the fourth moment

$$\begin{aligned}\mathbb{E}[A_{n,i}^4] =& \mathbb{E}[(B(t_i) - B(t_{i-1}))^8 - 4(B(t_i) - B(t_{i-1}))^6(t_i - t_{i-1}) + 6(B(t_i) \\ &- B(t_{i-1}))^4(t_i - t_{i-1})^2 - 4(B(t_i) - B(t_{i-1}))^2(t_i - t_{i-1})^3 + (t_i - t_{i-1})^4] \\ =& (105 - 60 + 18 - 4 + 1)(t_i - t_{i-1})^4 = 60(t_i - t_{i-1})^4\end{aligned}$$

where the expectation $\mathbb{E}[\cdot]$ is with respect to the probability space of Brownion motion, and we have used the moments $\mathbb{E}[Z^{2k}] = \sigma^{2k}(2k)!/(2^k k!)$ for $Z \sim \mathcal{N}(0, \sigma^2)$. Then we have that

$$\begin{aligned}\mathbb{E}[(Q(\Pi_n, B) - T)^4] =& \sum_i \mathbb{E}[A_{n,i}^4] + \sum_{i \neq j} \mathbb{E}[A_{n,i}^2]\mathbb{E}[A_{n,j}^2] \\ =& 60 \sum_i (t_i - t_{i-1})^4 + 4 \sum_{i \neq j} (t_i - t_{i-1})^2 (t_j - t_{j-1})^2 \\ \leq& 60 \sum_i (t_i - t_{i-1})^4 + 2 \sum_{i \neq j} \left((t_i - t_{i-1})^4 + (t_j - t_{j-1})^4\right) \\ \leq& (2n + 60) \sum_i (t_i - t_{i-1})^4\end{aligned}$$

Recall from the order statistics theory that $t_i - t_{i-1}$ follows a beta distribution with $\alpha = 1$ and $\beta = n$, or $Beta(1, n)$. The fourth moment of $Beta(1, n)$ is known to be

$$\frac{24}{(n+1)(n+2)(n+3)(n+4)}$$

Hence we have

$$\hat{\mathbb{E}}[(Q(\Pi_n, B) - T)^4] \leq (2n + 60) \sum_i (t_i - t_{i-1})^4 \leq \frac{24(2n + 60)n}{(n+1)(n+2)(n+3)(n+4)}$$

where $\hat{\mathbb{E}}[\cdot]$ is with respect to the probability space of both Brownian motion and uniform sampling. By Markov's inequality, we have

$$\begin{aligned}\mathbb{P}((Q(\Pi_n) - T)^4 \geq \epsilon) \leq& \frac{\hat{\mathbb{E}}[(Q(\Pi_n, B) - T)^4]}{\epsilon} \\ \leq& \frac{24(2n + 60)n}{\epsilon(n+1)(n+2)(n+3)(n+4)}\end{aligned}$$

which is summable across n. Borel-Cantelli lemma gives the conclusion. □

Problem 3.

By the definition of conditional expectation, it suffices to show that for every $B \in \mathcal{G}$, $\mathbb{E}[X\mathbb{1}_B] = \mathbb{E}[\mathbb{E}[X]\mathbb{1}_B]$. First, assume that $X \geq 0$. Observe that by the tail formula for expectation,

$$\begin{aligned}
\mathbb{E}[X\mathbb{1}_B] &= \int_0^\infty \mathbb{P}(X\mathbb{1}_B \geq y)dy \\
&= \int_0^\infty \mathbb{P}(\{X \geq y\} \cap B)dy \\
&= \int_0^\infty \mathbb{P}(\{X \geq y\})\mathbb{P}(B)dy && \text{by independence} \\
&= \mathbb{P}(B)\mathbb{E}[X] \\
&= \mathbb{E}[\mathbb{1}_B]\mathbb{E}[X] \\
&= \mathbb{E}[\mathbb{1}_B\mathbb{E}[X]].
\end{aligned}$$

In the general case,

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X^+ - X^-|\mathcal{G}] = \mathbb{E}[X^+|\mathcal{G}] - \mathbb{E}[X^-|\mathcal{G}] = \mathbb{E}[X^+] + \mathbb{E}[X^-] = \mathbb{E}[X],$$

giving the result.

Problem 4.

1. *Construct a function of the state $\varphi(x)$ for $x \in \mathbb{Z}$ such that $\varphi(Q(t))$ is a martingale.*

   Let $\varphi(x) = \left(\frac{1-p}{p}\right)^x$. Then, note that since φ is one-to-one, the event $\{Q(t) = z\}$ is the same as the event $\{\varphi(Q(t)) = \varphi(z)\}$. For some $z \in \mathbb{Z}$, it follows that

$$\begin{aligned}
\mathbb{E}[\varphi(Q(t+1)) \mid \varphi(Q(t)) = \varphi(z)] &= E[\varphi(Q(t+1) \mid Q(t) = z] \\
&= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{Q(t+1)} \middle| Q(t) = z\right] \\
&= p\left(\frac{1-p}{p}\right)^{z+1} + (1-p)\left(\frac{1-p}{p}\right)^{z-1} \\
&= (1-p)\left(\frac{1-p}{p}\right)^{z} + p\left(\frac{1-p}{p}\right)^{z} \\
&= \left(\frac{1-p}{p}\right)^{z} = \varphi(z) \\
&= \varphi(Q(t))
\end{aligned}$$

   thus $\varphi(Q(t))$ is a martingale.

2.

First, we claim that for every $i > z$, if B_i is the event that for all t, $0 < Q(t) < i$, $\mathbb{P}(B_i) = 0$. Suppose at time t we are in $(0, i)$. Then with at least probability $p^i > 0$, $Q(t+i) > i$, as this is the probability our walk increases in each of the i periods. For $j = 0, 1, \ldots$, let B_{ij} be the event that we remain between 0 and i on the time interval $[ij, i(j+1))$, so $B_i = \bigcup_j B_{ij}$. We compute that

$$\begin{aligned}
\mathbb{P}(B_i) &= \mathbb{P}\left(\bigcup_{j=0}^{\infty} B_{ij}\right) \\
&= \prod_{j=0}^{\infty} \mathbb{P}\left(B_{ij} \middle| \bigcap_{k=1}^{j-1} B_{ik}\right) \\
&\leq \prod_{j=0}^{\infty} (1 - p^i) \\
&= 0.
\end{aligned}$$

Thus for each i, with we will leave the interval $[0, i]$ almost surely. Let $\tau_i = \min\{t \in \mathbb{N} : Q(t) = i\}$, and for each $i > z$, let $T_i = \min\{\tau_i, \tau_0\}$. Each τ_i and each T_i are stopping times. Let $A_i = \{\tau_0 < \tau_i\}$, and $q_i = \mathbb{P}(A_i)$. Since $\varphi(Q(t \wedge T_i))$ is a bounded martingale, by considering the value at $t = T_i$, we obtain that

$$\varphi(z) = \mathbb{E}\left[\varphi\left(Q(0)\right)\right] = E\left[\varphi(Q(T_i)\right] = q_i\varphi(0) + (1 - q_i)\varphi(i)$$

yielding

$$q_i = \frac{\left(\frac{1-p}{p}\right)^z - \left(\frac{1-p}{p}\right)^i}{1 - \left(\frac{1-p}{p}\right)^i}.$$

Clearly the event that we ever hit zero, denoted by A, is given by

$$A = \bigcup_{i=1}^{\infty} A_i,$$

as if $\omega \in A$, then we hit zero at some finite time t, so we must hit zero before we hit $z+t$, and if $\omega \in A_i$ for some i, then as we hit zero before we hit i, we hit zero. Further, $A_i \subset A_{i+1}$ for all i, as we cannot hit $i+1$ until we hit i, so hitting zero before i forces us to hit zero before $i+1$ as well. Thus we can apply continuity from below to compute

$$\mathbb{P}(A) = \lim_{i\to\infty} \mathbb{P}(A_i) = \lim_{i\to\infty} \frac{\left(\frac{1-p}{p}\right)^z - \left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^i} = \left(\frac{1-p}{p}\right)^z,$$

as $p < 1/2$ implies $(1-p)/p < 1$. Therefore the probability that the random walk never hits zero is given by

$$1 - P(A) = 1 - \left(\frac{1-p}{p}\right)^z,$$

giving the result.

Problem 5.

1.

Hint: *Consider* $(X_j - X_{j-1})^2$.

As

$$0 \le (X_j - X_{j-1})^2 = X_j^2 - 2X_jX_{j-1} + X_{j-1}^2,$$

taking expectations we obtain that

$$\begin{aligned} 0 &\le \mathbb{E}[X_j^2] - 2\mathbb{E}[X_jX_{j-1}] + \mathbb{E}[X_{j-1}^2] \\ &= \mathbb{E}[X_j^2] - 2\mathbb{E}[\mathbb{E}[X_jX_{j-1}|\mathcal{F}_{j-1}]] + \mathbb{E}[X_{j-1}^2] \\ &= \mathbb{E}[X_j^2] - 2\mathbb{E}[X_{j-1}\mathbb{E}[X_j|\mathcal{F}_{j-1}]] + \mathbb{E}[X_{j-1}^2] \\ &= \mathbb{E}[X_j^2] - 2\mathbb{E}[X_{j-1}^2] + \mathbb{E}[X_{j-1}^2] \end{aligned}$$

and thus

$$\mathbb{E}[X_{j-1}^2] \le \mathbb{E}[X_j^2]$$

as claimed.

2.

If $X_n = X_1$ a.s., then $\mathrm{Var}(X_n - X_1) = 0$. From here, we compute that

$$0 = \mathrm{Var}(X_n - X_1) = \mathbb{E}[(X_n - X_1)^2] - \mathbb{E}[X_n - X_1]^2.$$

Furthermore, we have

$$\mathbb{E}[X_n - X_1]^2 = \mathbb{E}[\mathbb{E}[X_n - X_1|\mathcal{F}_{n-1}]]^2 = \mathbb{E}[X_{n-1} - X_1]^2,$$

and

$$\begin{aligned} \mathbb{E}[(X_n - X_1)^2] &= \mathbb{E}[\mathbb{E}[(X_n - X_1)^2|\mathcal{F}_{n-1}]] \\ &= \mathbb{E}[\mathbb{E}[X_n^2 - 2X_nX_1 + X_1^2|\mathcal{F}_{n-1}]] \\ &= \mathbb{E}[\mathbb{E}[X_n^2|\mathcal{F}_{n-1}]] + \mathbb{E}[-2X_{n-1}X_1 + X_1^2] \\ &= \mathbb{E}[X_n^2] + \mathbb{E}[-2X_{n-1}X_1 + X_1^2] \\ &\ge \mathbb{E}[X_{n-1}^2] + \mathbb{E}[-2X_{n-1}X_1 + X_1^2] \\ &= \mathbb{E}[X_{n-1}^2 - 2X_{n-1}X_1 + X_1^2] \\ &= \mathbb{E}[(X_{n-1} - X_1)^2]. \end{aligned}$$

Thus

$$0 = \mathbb{E}[(X_n - X_1)^2] - \mathbb{E}[X_n - X_1]^2 \geq \mathbb{E}[(X_{n-1} - X_1)^2] - \mathbb{E}[X_{n-1} - X_1]^2 = \text{Var}(X_{n-1} - X_1).$$

As variances are nonnegative, $\text{Var}(X_{n-1} - X_1) = 0$, so $X_{n-1} = X_1 = X_n$ a.s.. The result follows by induction.

Problem 6.

a).

Suppose there exists a countably infinite strictly increasing sequence $t_n \in \mathbb{R}_+$, $n \geq 0$ such that

$$\mathbb{P}(T \in \{t_n : n \in \mathbb{N}\} \cup \{\infty\}) = 1.$$

Emulate the proof of the discrete time processes to show that $X_{t\wedge T}$, for $t \in \mathbb{R}_+$ is a submartingale .

Fix arbitrary $0 < s < t$, and let n_1 and n_2 be positive integers such that

$$t_{n_1} \leq t < t_{n_1+1}$$

and

$$t_{n_2} - 1 \leq s < t_{n_2}$$

which implies $t_{n_2} \leq t_{n_1}$. We want to show that $\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] \geq X_{s\wedge T}$.

First, using the exact same proof as in the lecture notes, we can show that the sequence $Y_n = X_{t_n\wedge T}$ is a submartingale with respect to the sequence $\mathcal{F}'_n = \mathcal{F}_{t_n}$. Briefly, the sequence $H_n = \{T \geq t_n\}$ is predictable with respect to $\mathcal{F}'_n$, so that the the sequence $\sum_{m\leq n} H_m(X_{t_{m+1}} - X_{t_m})$ is a martingale, and this sequence is equal to $-X_{t_0} + X_{t_n\wedge T}$, which gives us the desired result. Next, we show the following two inequalities:

$$\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_{t_{n_1}}] \geq X_{t_{n_1}\wedge T}$$

and

$$\mathbb{E}[X_{t_{n_2}\wedge T} \mid F_s] \geq X_{s\wedge T}$$

For the first, write

$$\begin{aligned}
\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_{t_{n_1}}] =& \mathbb{E}[(1_{T>t_{n_1}} + 1_{T\leq t_{n_1}})X_{t\wedge T} \mid \mathcal{F}_{t_{n_1}}] \\
=& \mathbb{E}[1_{T>t_{n_1}} X_t + 1_{T\leq t_{n_1}} X_T \mid \mathcal{F}_{t_{n_1}}] \\
=& \mathbb{E}[1_{T>t_{n_1}} X_t \mid \mathcal{F}_{t_{n_1}}] + \mathbb{E}[1_{T\leq t_{n_1}} X_T \mid \mathcal{F}_{t_{n_1}}] \\
=& \mathbb{E}[1_{T>t_{n_1}} E[X_t \mid \mathcal{F}_{t_{n_1}}] \mid \mathcal{F}_{t_{n_1}}] + \mathbb{E}[1_{T\leq t_{n_1}} X_T \mid \mathcal{F}_{t_{n_1}}] \\
\geq& \mathbb{E}[1_{T>t_{n_1}} X_{t_{n_1}} \mid \mathcal{F}_{t_{n_1}}] + \mathbb{E}[1_{T\leq t_{n_1}} X_T \mid \mathcal{F}_{t_{n_1}}] \\
\geq& \mathbb{E}[X_{t_{n_1}\wedge T} \mid \mathcal{F}_{t_{n_1}}] = X_{t_{n_1}\wedge T}
\end{aligned}$$

The second equality comes from the fact that if $T > t_{n_1}$, then we necessarily have $T \geq t_{n_1+1} > t$. Third equality comes from linearity of expectations. The fourth comes from the definition of conditional expectation. The first inequality comes from the fact that X_t is a martingale. The second comes from the fact that the sequence $X_{t_n\wedge T}$ is a martingale w.r.t. $\mathcal{F}'_n$. The inequality $\mathbb{E}[X_{t_{n_2}\wedge T} \mid F_s] \geq X_{s\wedge T}$ is shown similarly. Finally, we simply use the tower property repeatedly and our two inequalities:

$$\begin{aligned}
\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] =& \mathbb{E}[\mathbb{E}[X_{t\wedge T} \mid F_{t_{n_1}}] \mid \mathcal{F}_s] \\
\geq& \mathbb{E}[X_{t_{n_1}\wedge T} \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X_{t_{n_1}\wedge T} \mid t_{n2}] \mid \mathcal{F}_s] \\
\geq& \mathbb{E}[X_{t_{n_2}\wedge T} \mid \mathcal{F}_s] \geq X_{s\wedge T}
\end{aligned}$$

QED

b).

Given a general stopping time T taking values in $\mathbb{R}_+ \cup \{\infty\}$, consider a sequence of random variables T_n defined by $T_n(\omega) = k/2^n$ for $k = 1, 2, \dots,$ if $T(\omega) \in [(k-1)/2^n, k/2^n)$, and $T_n(\omega) = \infty$ if $T(\omega) = \infty$. Establish that T_n is a stopping time for every n.

Let $t_{n,k} = \frac{k}{2^n}$. The event $\{T_n \leq t_{n,k}\}$ is equal to the event $\{T < t_{n,k}\} = \cup_n \{T \leq t_{n,k} - 1/n\}$. Since T is a stopping time, each event $\{T \leq t_{n,k} - 1/n\}$ is measurable w.r.t $\mathcal{F}_{t_{n,k}}$, and $\{T_n \leq t_{n,k}\}$ is therefore measurable with respect to $\mathcal{F}_{t_{n,k}}$. This proves that T_n is a stopping time.

c).

Suppose the submartingale X_t is in $\mathbb{L}_2$, namely $\mathbb{E}[X^2] < \infty$ for all t. Show that $X_{T\wedge t}$ is a submartingale as well.

Hint: *Use part 5b, the Doob-Kolmogorov inequality, and the Dominated Convergence theorem.*r

For any real number x, let $C(x)$ be the closest integer strictly greater than x ($C(1.1) = 2, C(2) = 3$). With this definition, it is easy to show that $T_n = \frac{C(2^n T)}{2^n}$. Note that $T_{n+1} - T_n = \frac{C(2^{n+1}T) - 2C(2^n T)}{2^{n+1}} \leq 0$, so that T_n decreases. Moreover, $|T_n - T| \leq \frac{1}{2^n}$, so that T_n converges surely to T. Let $s < t$ be two positive real.

$$\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] = \mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] - \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s] + \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s]$$

By part a, we obtain

$$\begin{aligned}\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] &\geq \mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] - \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s] + X_{s\wedge T_n} \\ &\geq \mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] - \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s] + X_{s\wedge T_n} - X_{s\wedge T} + X_{s\wedge T}\end{aligned}$$

Because X is RCLL and T_n decreases to T , we obtain that $X_{s\wedge T_n}$ converges to $X_{s\wedge T}$. Therefore, if we can show that $\lim_n \mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] - \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s]$ converges to 0, by taking limits we will obtain that $\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] = X_{s\wedge T}$, and we will be done. Note that since T_n is decreasing, $|X_{t\wedge T_n}|$ is upper bounded by $\sup_{[0,t]} X_t$. By Doob-Kolmogorov, $\mathbb{P}(\sup_{[0,t]} X_t \geq \varepsilon) \leq \frac{E[X_T^2]}{\varepsilon^2}$. We conclude that $E[\sup_{[0,t]} |X_t|]$ is finite; and by the conditional dominated convergence theorem, $\lim_n \mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] - \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s]$, and so, $\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] = X_{s\wedge T}$, QED.

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