MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Homework 4 Solution

1 Problem 1

Proof. We aim to bound $|\mathbb{E}[L_n] - m_n|$ by $O(\sqrt{m_n})$. We have that

$$|\mathbb{E}[L_n] - m_n| \leq \mathbb{E}[|L_n - m_n|] = \int_0^\infty \mathbb{P}(|L_n - m_n| > x)dx$$

By Talagrand's inequality, we have that

$$\mathbb{P}(|L_n - m_n| > x) \leq 4\exp(-\frac{x^2}{4(m_n + x)})$$

Let $x = \beta\sqrt{m_n \log m_n}$, we have

$$\mathbb{P}(|L_n - m_n| > x) \leq 4\exp(-\frac{\beta^2 m_n \log m_n}{4(m_n + \beta\sqrt{m_n \log m_n})})$$

Let $\beta_0 = c\sqrt{\frac{m_n}{\log m_n}}$ for some $c \geq 1$, we have for $\beta \leq \beta_0$

$$\mathbb{P}(|L_n - m_n| > x) \leq 4\exp(-\frac{\beta^2 \log m_n}{4(1+c)}) = 4m_n^{-\frac{\beta^2}{4(1+c)}}$$

For $\beta > \beta_0$, let $\beta = y\sqrt{m_n}\log m_n$ where $y > c \geq 1$, and then

$$\mathbb{P}(|L_n - m_n| \geq \beta\sqrt{m_n \log m_n}) \leq 4\exp(-\frac{y^2 m_n^2}{4(m_n + y m_n)}) = 4\exp(-\frac{y^2 m_n}{4(1+y)})$$
$$\leq 4\exp(-\frac{y m_n}{8})$$

Thus,

$$\int_0^\infty \mathbb{P}(|L_n - m_n| > x)dx \leq \sqrt{m_n \log m_n}\int_0^{\beta_0} 4m_n^{-\frac{\beta^2}{4(1+c)}}d\beta + m_n\int_c^\infty 4\exp(-\frac{y m_n}{8})dy$$
$$= 4\sqrt{m_n \log m_n}\sqrt{\frac{\pi(1+c)}{\log m_n}} + 4m_n\frac{8\exp(-\frac{c m_n}{8})}{m_n}$$
$$\leq 4\sqrt{\pi(1+c)m_n} + 32 = O(m_n)$$

Namely,

$$|\mathbb{E}[L_n] - m_n| \leq O(\sqrt{m_n})$$

Thus, we have $m_n = \alpha\sqrt{n} + o(\sqrt{n})$. ☐

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2 Problem 2 (Kaji's Solution)

Proof. Define g by $g(E) = g(E_1, ..., E_{dn}) = \log Z_n$. For notational purposes, denote $E_{-i} := (E_1, ..., E_{i-1}, E_{i+1}, ..., E_{dn})$. Then we will prove the following equation:

$$|g(E, E_{-i} - g(E_i^{'}, E_{-i}))| \leq \log 2 \text{ for all } i. \tag{1}$$

Consider the change from $g(E_i, E_{-i})$ to $g(E_{-i})$. Omitting the edge E_i, we have that the sets that are independent under E are still independent under E_{-i}, and that for an independent set I under E, a new set $I \cup E_i$ may become independent. Hence, the maximum possible change is $Z_n \to 2Z_n$. Now, by adding a new edge $E_i^{'} = \{a, b\}$, observe the following:

- The independent sets that do not contain $E_i^{'}$ are still independent.
- If I is an independent set containing $E_i^{'}$, then $I \backslash \{a\}$ and $I \backslash \{b\}$ are still independent.
- So the number of independent sets eliminated by adding $E_i^{'}$ cannot exceed $\frac{1}{3}$ of that of independent sets under E_{-i}.

Thus, $Z_n^{'}$ cannot go below $\frac{2}{3} Z_n$, and we have (1). Let $d_i = \log 2$. Since g is symmetric in i, Theorem 12.2 gives that for any $t > 0$,

$$\mathbb{P}(|g(E) - \mathbb{E}[g(E)]| \geq t) \leq 2\exp(-\frac{t^2}{2dn(\log 2)^2})$$

Thus, we have

$$\mathbb{P}(\log Z_n - \mathbb{E}[\log Z_n] \geq t) \leq \mathbb{P}(|g(E) - \mathbb{E}[g(E)]| \geq t) \leq 2\exp(-\frac{t^2}{2dn(\log 2)^2})$$

□

3 Problem 3

Proof. Let $X_t = X_t^+ - X_t^-$ where $X_t^+ = \max\{X_t, 0\}$ and $X_t^- = \max\{-X_t, 0\}$, then we have that

$$X_t = X_t^+ - X_t^-$$

Similar to the proof of Proposition 3, define two increasing nonnegative sequences X_t^{n+} and X_t^{n-}: $X_t^{n+} = X_t^+$ when $0 \leq X_t^+ \leq n$ and $X_t^{n+} = n$

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o.w.; $X_t^{n-} = X_t^-$ when $0 \leq X_t^- \leq n$ and $X_t^{n-} = n$ o.w.. Thus, a.s. we have that

$$X_t^{n+} \to X_t^+, \; X_t^{n-} \to X_t^-$$

Let $X_n = X_t^{n+} - X_t^{n-}$. By $|X_n| \leq |X|$ and $|X_n| \leq n$, we have that X_n is a.s. bounded and in $\mathcal{L}_2$. Then,

$$\begin{aligned}(X_t - X_t^n)^2 &= ((X_t^+ - X_t^{n+}) - (X_t^- - X_t^{n-}))^2 \\ &\leq 2(X_t^+ - X_t^{n+})^2 + 2(X_t^- - X_t^{n-})^2 \end{aligned} \tag{2}$$

We have that

$$\mathbb{E}[\int_0^T (X_t^+ - X_t^{n+})^2 dt] = \mathbb{E}[\int_0^T (X_t^+)^2 dt] + \mathbb{E}[\int_0^T (X_t^{n+})^2 dt] - 2\mathbb{E}[\int_0^T X_t^+ X_t^{n+} dt]$$

As $n \to \infty$, by MCT the r.h.s. of the equation above converges to

$$\mathbb{E}[\int_0^T (X_t^+)^2 dt] + \mathbb{E}[\int_0^T (X_t^+)^2 dt] - 2\mathbb{E}[\int_0^T (X_t^+)^2 dt] = 0$$

Likewise, we have that $\mathbb{E}[\int_0^T (X_t^- - X_t^{n-})^2 dt] \to 0$ as $n \to \infty$. By (2), we have that

$$\begin{aligned}\mathbb{E}[\int_0^T (X_t - X_t^n)^2 dt] &\leq 2\mathbb{E}[\int_0^T (X_t^+ - X_t^{n+})^2 dt] + \mathbb{E}[\int_0^T (X_t^- - X_t^{n-})^2 dt] \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

□

4 Problem 4

Proof. Since $\mathbb{E}[\int_0^t (X_s^n - X_s)^2 ds] \to 0$ as $n \to \infty$, given $\epsilon > 0$, there exists a positive integer N such that for any $n > N$ we have

$$\mathbb{E}[\int_0^t (X_s^n - X_s)^2 ds] < \epsilon$$

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For any $n > N$, we have that

$$\begin{aligned}
\mathbb{E}[\int_0^t (X_s^n + X_s)^2 ds] &= \mathbb{E}[\int_0^t (X_s^n - X_s + 2X_s)^2 ds] \\
&= \mathbb{E}[\int_0^t (X_s^n - X_s)^2 + 4X_s(X_s^n - X_s) + 4X_s^2 ds] \\
&\leq \mathbb{E}[\int_0^t 2(X_s^n - X_s)^2 + 8X_s^2 ds] \\
&\leq 2\epsilon + 8\mathbb{E}[\int_0^t X_s^2 ds] < \infty \qquad (3)
\end{aligned}$$

For any $n \leq N$, we have that

$$\begin{aligned}
\mathbb{E}[\int_0^t (X_s^n + X_s)^2 ds] &\leq 2\mathbb{E}[\int_0^t (X_s^n)^2 ds] + 2\mathbb{E}[\int_0^t X_s^2 ds] \\
&< \infty \qquad (4)
\end{aligned}$$

Thus, (3) and (4) give that

$$\sup_n \mathbb{E}[\int_0^t (X_s^n + X_s)^2 ds] < \infty$$

□

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Problem 5.

The goal of this exercise is to show that much of Ito Calculus can be generalized to integration with respect to arbitrary continuous square integrable martingales.

a. *Define $\int_0^t X_s dM_s$ for simple processes. Show that the resulting process is a martingale and establish Ito isometry for it.*

As X_t is simple, for every T there exists finitely many $0 = t_0 < t_1 < \cdots < t_n = T$ such that for all $t_j \le s < t_j$, $X_s = X_{t_j}$. Assume for simplicity that $t = t_n$ for some n, and define

$$\int_0^s X_s dM_s = \sum_{i \le n-1} X_{t_i}(M_{t_{i+1}} - M_{t_i})$$

First, we show $\int_0^T X_t dM_t$ is a martingale. Fix $0 \le s < T$; there exists k such that $t_k \le s < t_{k+1}$.

$$\begin{aligned}
\mathbb{E}\left[\int_0^T X_t dM_t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[\sum_{j=0}^{n-1} X_{t_j}(M_{t_{j+1}} - M_{t_j}) \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{k-1} X_{t_j}(M_{t_{j+1}} - M_{t_j}) \middle| \mathcal{F}_s\right] + \mathbb{E}\left[X_{t_k}(M_{t_{k+1}} - M_{t_k}) \middle| \mathcal{F}_s\right] \\
&\quad + \mathbb{E}\left[\sum_{j=k+1}^{n-1} X_{t_j}(M_{t_{j+1}} - M_{t_j}) \middle| \mathcal{F}_s\right] \\
&= \sum_{j=0}^{k-1} X_{t_j}(M_{t_{j+1}} - M_{t_j}) + \mathbb{E}\left[X_k(M_{t_{k+1}} - M_{t_k}) \middle| \mathcal{F}_s\right] \\
&\quad + \mathbb{E}\left[\sum_{j=k+1}^{n-1} X_{t_j}(M_{t_{j+1}} - M_{t_j}) \middle| \mathcal{F}_s\right].
\end{aligned}$$

For $j \ge k+1$,

$$\mathbb{E}[X_{t_j}(M_{t_{j+1}} - M_{t_j})] = \mathbb{E}[X_{t_j}\mathbb{E}[M_{t_{j+1}} - M_{t_j}|\mathcal{F}_{t_j}]|\mathcal{F}_s] = 0$$

For the middle term,

$$\begin{aligned}
\mathbb{E}\left[X_{t_k}(M_{t_{k+1}} - M_{t_k}) \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[X_{t_k}(M_{t_{k+1}} - M_s)|\mathcal{F}_s\right] + \mathbb{E}\left[X_{t_k}(M_s - M_{t_k}) \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[X_{t_k}(M_{t_{k+1}} - M_s)|\mathcal{F}_s\right] + \mathbb{E}\left[X_{t_k}(M_s - M_{t_k}) \middle| \mathcal{F}_s\right] \\
&= X_{t_k}\left(\mathbb{E}\left[M_{t_{k+1}} \middle| \mathcal{F}_s\right] - M_s\right) + X_{t_k}(M_s - M_{t_k}) \\
&= X_{t_k}(M_s - M_{t_k}),
\end{aligned}$$

as M_t is a martingale, and we conclude that the integral is a Martingale. Next we show the Ito Isometry property. Let

$$\Delta\langle M_{t_j}\rangle \triangleq \langle M_{t_{j+1}}\rangle - \langle M_{t_j}\rangle.$$

We compute that

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^T X_t dM_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{j=0}^{n-1} X_{t_j}(M_{t_{j+1}} - M_{t_j})\right)^2\right] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \mathbb{E}\left[X_{t_i}X_{t_j}(M_{t_{i+1}} - M_{t_i})(M_{t_{j+1}} - M_{t_j})\right] \\
&= \sum_{j=0}^{n-1} \mathbb{E}\left[X_{t_j}^2(M_{t_{j+1}} - M_{t_j})^2\right] + 2\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} \mathbb{E}\left[X_{t_i}X_{t_j}(M_{t_{i+1}} - M_{t_i})(M_{t_{j+1}} - M_{t_j})\right].
\end{aligned}
$$

But for each term in the second sum, as $i > j$, and thus $t_i > t_j$,

$$
\begin{aligned}
\mathbb{E}\left[X_{t_i}X_{t_j}(M_{t_{i+1}} - M_{t_i})(M_{t_{j+1}} - M_{t_j})\right] &= \mathbb{E}\left[\mathbb{E}\left[X_{t_i}X_{t_j}(M_{t_{i+1}} - M_{t_i})(M_{t_{j+1}} - M_{t_j})\middle|\mathcal{F}_{t_i}\right]\right] \\
&= \mathbb{E}\left[X_{t_i}X_{t_j}(\mathbb{E}[[M_{t_{i+1}}|\mathcal{F}_{t_i}] - M_{t_i})(M_{t_{j+1}} - M_{t_j})\right] \\
&= 0,
\end{aligned}
$$

as the remaining terms are all finite in expectation. Continuing, with the tower property again

$$
\begin{aligned}
\sum_{j=0}^{n-1} \mathbb{E}\left[X_{t_j}^2(M_{t_{j+1}} - M_{t_j})^2\right] &= \sum_{j=0}^{n-1} \mathbb{E}\left[X_{t_j}^2 \mathbb{E}\left[(M_{t_{j+1}} - M_{t_j})^2\middle|\mathcal{F}_{t_j}\right]\right] \\
&= \sum_{j=0}^{n-1} \mathbb{E}\left[X_{t_j}^2 \Delta\langle M_{t_j}\rangle\right] \\
&= \mathbb{E}\left[\int_0^T X_t^2 d\langle M_t\rangle\right],
\end{aligned}
$$

b. *Given an arbitrary $X \in \mathbb{L}_2$, show that if X^n is a sequence of processes such that $\lim_n \mathbb{E}\int_0^t (X_s^n - X_s)^2 d\langle M_s\rangle = 0$, then the sequence $\int_0^t X_s^n dM_s$ is Cauchy for every t in the $\mathbb{L}_2$ sense. Use this to define $\int_0^t X_s dM_s$ for any process $X \in \mathbb{L}_2$ and establish Ito isometry for the Ito integral. Here the integration $d\langle M_s\rangle$ is understood in the Stieltjes sense. You do not need to prove the existence of the processes X_t^n satisfying the requirement above (unless you would like to).*

To show $\int_0^t X_s^n dM_s$ is Cauchy in $\mathbb{L}_2$, we need for every ε there exists n such that for all $m > n$,

$$\mathbb{E}\left[\left(\int_0^t X_s^n dM_s - \int_0^t X_s^m dM_s\right)^2\right] \leq \varepsilon.$$

By the linearity of the Ito integral for simple processes (same proof as the linearity of Ito integral w.r.t Brownian motion), and noting that the difference of two simple processes is a simple processes, applying Ito Isometry gives

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t X_s^n dM_s - \int_0^t X_s^m dM_s\right)^2\right] &= \mathbb{E}\left[\left(\int_0^t X_s^n - X_s^m dM_s\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t (X_s^n - X_s^m)^2 d\langle M_s\rangle\right] \\
&= \mathbb{E}\left[\int_0^t (X_s^n - X_s + X_s - X_s^m)^2 d\langle M_s\rangle\right] \\
&\leq \mathbb{E}\left[\int_0^t 2(X_s^n - X_s)^2 + 2(X_s - X_s^m)^2 d\langle M_s\rangle\right].
\end{aligned}$$

Recalling back to how we defined the above integral in the final equality of **3a**, we see that the integral is linear, and as expectation is as well, we obtain

$$\mathbb{E}\left[\int_0^t 2(X_s^n - X_s)^2 + 2(X_s - X_s^m)^2 d\langle M_s\rangle\right] = 2\mathbb{E}\left[\int_0^t (X_s^n - X_s)^2 d\langle M_s\rangle\right] + 2\mathbb{E}\left[\int_0^t (X_s - X_s^m)^2 d\langle M_s\rangle\right].$$

By assumption, as $\lim_n \mathbb{E}\int_0^t (X_s^n - X_s)^2 d\langle M_s\rangle = 0$, if we choose n sufficiently large,

$$2\mathbb{E}\left[\int_0^t (X_s^n - X_s)^2 d\langle M_s\rangle\right] + 2\mathbb{E}\left[\int_0^t (X_s - X_s^m)^2 d\langle M_s\rangle\right] \leq 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon,$$

establishing the sequence is Cauchy. Since $\mathcal{M}_{2,C}$ is complete, there exists a limit, which we define as the Ito integral. The limit of the sequence is a square integrable martingale as well, and a very simple proof shows that the limit does not depend on the sequence X_s^n, making the Ito integral well defined. By taking limits, the Ito integral is linear.

We prove the Ito Isometry:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T X_t dM_t\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T (X_t - X_t^n) dM_t + \int_0^T X_t^n dM_t\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^T (X_t - X_t^n) dM_t\right)^2\right] \\
&\quad + 2\mathbb{E}\left[\int_0^T (X_t - X_t^n) dM_t \int_0^T X_t^n dM_t\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^T X_t^n dM_t\right)^2\right]
\end{aligned}$$

Passing the limit and following the same steps as the lecture notes, we obtain that the Ito isometry holds in the limit.

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