Geometry

Warm-up: A Theorem by Edmonds and Giles

Theorem 1 (Edmonds and Giles 1977)**.** *A rational polyhedron P is integral if and only if for all integral vectors w the optimal value of* $\max\{wx : x \in P\}$ *is integer.*

Proof (for polytopes):

- Let v be a vertex of P, and let $w \in \mathbb{Z}^n$ be such that v is the unique optimal solution to $\max\{wx : x \in P\}$.
- By multiplying w by a large positive integer if necessary, we may assume that $wv > wu + u_1 - v_1$ for all vertices u of P other than v.
- If we let $\overline{w} := (w_1 + 1, w_2, \dots, w_n)$, then v is an optimal solution to $\max\{\overline{w}x : x \in P\}$.
- So $\overline{w}v, = wv + v_1$, and both $\overline{w}v$ and wv are integer.
- Thus v_1 is an integer.
- Repeat for the remaining components of v. □

Totally Unimodular Matrices

- Recall that *totally unimodular* matrices are exactly those integral matrices A for which the polyhedron $\{x \geq 0 : Ax \leq b\}$ is integral for each integral vector b.
- This concept has led to a number of important results by virtue of the LP-duality equation

$$\max\{wx : x \geq 0, Ax \leq b\} = \min\{yb : yA \geq w\}.$$

For instance, ...

- König's Theorem: The maximum cardinality of a stable set in a bipartite graph is equal to the minimum number of edges needed to cover all nodes.
- König-Egerváry's Theorem: The maximum cardinality of a matching in a bipartite graph is equal to the minimum cardinality of a set of nodes intersecting each edge.
- The Max-Flow-Min-Cut Theorem: The maximum value of an s-t flow is equal to the minimum capacity of any s-t cut.
- ...

Total Dual Integrality

- In this lecture, we fix A *and* b and study integer polyhedra.

Consider the LP-duality equation

$$\max\{wx : Ax \leq b\} = \min\{yb : yA = w, y \geq 0\}.$$

- If b is integral and the minimum has an integral optimal solution y for each integral vector w, then the maximum also has an integral optimum solution, for each such w.

- A rational system $Ax \leq b$ is called *totally dual integral* (TDI) if the minimum has an integral optimum solution y for each integral vector w (for which the optimum is finite).

- Thus, if $Ax \leq b$ is TDI and b is integral, then $P = \{x : Ax \leq b\}$ is integral.

Total dual integrality is *not* a property of polyhedra. The systems

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

define the same polyhedron, but the second one is TDI, whereas the first one is not.

TDI Representations

Theorem 2 (Giles and Pulleyblank 1979)**.** *Let P be a rational polyhedron. There exists a totally dual integral system $Ax \leq b$, with A integral, such that $P = \{x : Ax \leq b\}$. Furthermore, if P is an integral polyhedron, then b can be chosen to be integral.*

Integral Hilbert Bases
Let C be a rational polyhedral cone. A set of integral vectors $\{a_1, \ldots, a_t\}$ is an *integral Hilbert basis* of C if each integral vector in C is a nonnegative integral combination of $a_1, \ldots, a_t$.

Theorem 3. *Each rational polyhedral cone C is generated by an integral Hilbert basis. (If C is pointed, there is a unique minimal integral Hilbert basis.)*

Integral Hilbert Bases
Proof:

- Let $c_1, \ldots, c_k$ be primitive integral vectors that generate C.

- Consider $Z := \{\lambda_1 c_1 + \cdots + \lambda_k c_k : 0 \leq \lambda_i \leq 1\}$.

- Let H be the set of integral vectors in Z. Claim: H is a Hilbert basis.

- Let $c \in C \cap \mathbb{Z}^n$. Then $c = \lambda_1 c_1 + \cdots + \lambda_k c_k$, where $\lambda_i \geq 0$ for all i.

- Rewrite as $c - \sum_{i=1}^{k} \lfloor \lambda_i \rfloor c_i = \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) c_i$.

- Since the LHS is integral, so is the RHS. However, the RHS belongs to Z.

- So c is nonnegative integer combination of elements in H. □

Hilbert Bases and TDI Systems

Lemma 4. *$Ax \leq b$ is TDI if and only if for each minimal face F of $P = \{x : Ax \leq b\}$ the rows of A which are active in F form a Hilbert basis.*

Proof:

- Assume that $Ax \leq b$ is TDI. Let $a_1, \ldots, a_t$ be the rows of A active in F.

- Let c be an integral vector in $\text{cone}\{a_1, \ldots, a_t\}$.

- The maximum of
$$\max\{cx : Ax \leq b\} = \min\{yb : yA = c, y \geq 0\} \tag{1}$$
is attained by each vector x in F. The minimum has an integral optimal solution y.

- y has 0's in positions corresponding to rows not active in F.

- Hence, c is an integral nonnegative combination of $a_1, \ldots, a_t$.

- For the other direction, let $c \in \mathbb{Z}^n$ be such that the optima in (1) are finite.

- Let F be a minimal face of P so that each vector in F attains the maximum in (1).

- Let $a_1, \ldots, a_t$ be the rows active in F.

- Then $c \in \text{cone}\{a_1, \ldots, a_t\}$.

- In particular, $c = \lambda_1 a_1 + \cdots + \lambda_t a_t$ for certain $\lambda_i \in \mathbb{Z}_+$.

- Extending $(\lambda_1, \ldots, \lambda_t)$ with 0's, we obtain an integral vector $y \geq 0$ such that $yA = c$ and $yb = yAx = cx$ for all $x \in F$.

- So y attains the minimum in (1) □

TDI Representations

Theorem 5 (Giles and Pulleyblank 1979)**.** *Let P be a rational polyhedron. There exists a totally dual integral system $Ax \le b$, with A integral, such that $P = \{x : Ax \le b\}$. Furthermore, if P is an integral polyhedron, then b can be chosen to be integral.*

Proof:

- Let F be a minimal face of P.
- Let C_F be the normal cone of F.
- Let $a_1, \ldots, a_t$ be an integral Hilbert basis for C_F.
- For some $x_0 \in F$, let $b_i := a_i x_0$.
- The system Σ_F of inequalities

  $$a_1 x \le b_1, \ldots, a_t x \le b_t$$

  is valid for P.
- Let $Ax \le b$ be the union of all Σ_F over all minimal faces F.
- $Ax \le b$ defines P and is TDI.
- And if P is integral, then so is b. □

Procedure for Proving Integrality of Polyhedra

- Find an appropriate defining system $Ax \le b$, with A and b integral.
- Prove that $Ax \le b$ is totally dual integral.
- Conclude that $\{x : Ax \le b\}$ is an integral polyhedron.

An Application of Total Dual Integrality

Recall that, if $(N, \mathcal{I})$ is a matroid, then the convex hull of incidence vectors is equal to

$$P_{\mathcal{I}} = \text{conv}\{x \in \mathbb{R}^N_+ : x(S) \le r(S) \text{ for all } S \subseteq N\},$$

where r is the rank function of the matroid.

Theorem 6 (Matroid Intersection Theorem)**.** *The convex hull of the characteristic vectors of common independent sets of two matroids $(N, \mathcal{I}_1)$ and $(N, \mathcal{I}_2)$ is precisely the set of feasible solutions to*

$$\begin{aligned} x(S) &\le r_1(S) && \textit{for all } S \subseteq N \\ x(S) &\le r_2(S) && \textit{for all } S \subseteq N \\ x_j &\ge 0 && \textit{for all } j \in N. \end{aligned}$$

Proof:

- We show that the system is TDI.
- Consider the dual of maximizing wx over it:

$$\begin{aligned} \min \sum_{S \subseteq N} & (r_1(S)y_S^1 + r_2(S)y_S^2) \\ \text{s.t.} \sum_{N \supseteq S \ni j} & (y_S^1 + y_S^2) \geq w_j \text{ for } j \in N \\ & y_S^1, y_S^2 \geq 0 \text{ for } S \subseteq N \end{aligned}$$

- Let (y^1, y^2) be an optimal solution such that

$$\sum_{S \subseteq N} (y_S^1 + y_S^2))|S||N \setminus S| \tag{2}$$

  is minimized.

- Let $\mathcal{F}_i := \{S \subseteq N : y_S^i > 0\}$, for $i = 1, 2$.
- Claim: If $S, T \in \mathcal{F}_i$, then $S \subseteq T$ or $T \subseteq S$.
- Suppose not. Choose $\alpha := \min\{y_S^i, y_T^i\}$.
- Decrease y_S^i and y_T^i by α, and increase $y_{S\cap T}^i$ and $y_{S\cup T}^i$ by α.
- Since $\chi^S + \chi^T = \chi^{S\cap T} + \chi^{S\cup T}$, (y^1, y^2) remains feasible.
- Since $r_i(S) + r_i(T) \geq r_i(S \cap T) + r_i(S \cup T)$, it remains optimal.
- However, (2) decreases, contradicting the minimality assumption.
- The constraints corresponding to $\mathcal{F}_1$ and $\mathcal{F}_2$ form a totally unimodular matrix. □

Corollary 7.

$$P(\mathcal{I}_1 \cap \mathcal{I}_2) = P(\mathcal{I}_1) \cap P(\mathcal{I}_2)$$

An Application of Hilbert bases

- Consider $\max\{wx : Ax = b, 0 \leq x \leq u\}$.
- For $j = 1, \ldots, 2^n$, let O_j be the j-th orthant of $\mathbb{R}^n$.
- Let $C^j := \{x \in O_j : Ax = 0\}$, and let H^j be an integral Hilbert basis of C^j.

Theorem 8 (Graver 1975)**.** *A feasible solution x is optimal if and only if for every $h \in \bigcup_{j=1}^{2^n} H^j$ the following holds:*

1. *$wh \leq 0$, or*
2. *$wh > 0$ and $x + h$ is infeasible.*

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