

# 15.433 INVESTMENTS

Classes 6: The CAPM and APT

Part 1: Theory

Spring 2003

# Introduction

So far, we took the expected return of risky asset as given. But where does *expected return* come from?

Using the intuition that investors are risk averse, one explanation is that the *risk premium* - expected return in excess of the riskfree rate - is a reward for bearing risk.

Does this make sense?

The Capital Asset Pricing Model (CAPM) provides a simple, yet elegant framework for us to think about the question of *reward and risk*.

## ”The CAPM ”

In market equilibrium, investors are only rewarded for bearing systematic risk - the type of risk that cannot be diversified away.

They should not be rewarded for bearing idiosyncratic risk, since this uncertainty can be mitigated through appropriate diversification.

## Sharpe on CAPM

Bill Sharpe, one of the originators of the CAPM, in an interview with the Dow Jones Asset Manager:

*”But the fundamental idea remains that there’s no reason to expect reward just for bearing risk. Otherwise, you’d make a lot of money in Las Vegas. If there’s reward for risk, it’s got to be special. There’s got to be some economics behind it or else the world is a very crazy place. I don’t think differently about those basic ideas at all”.*

- Sharpe (1998)

# Assumptions

## 1. Perfect Markets

- Perfect competition - each investor assumes he has no effect on security prices
- No taxes
- No transactions costs
- All assets publicly traded, perfectly divisible
- No short-sale constraints
- Same riskfree rate for borrowing and lending

## 2. Identical Investors

- Myopic<sup>1</sup>
- Same holding period
- Normality or Mean-Variance Utility
- Homogeneous expectations

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<sup>1</sup>myopic: adj. Of, pertaining to, or affected with myopia; short-sighted, near-sighted

# The Equilibrium Market Portfolio

Recall that every investor holds some combination of the riskless asset and the tangency portfolio.

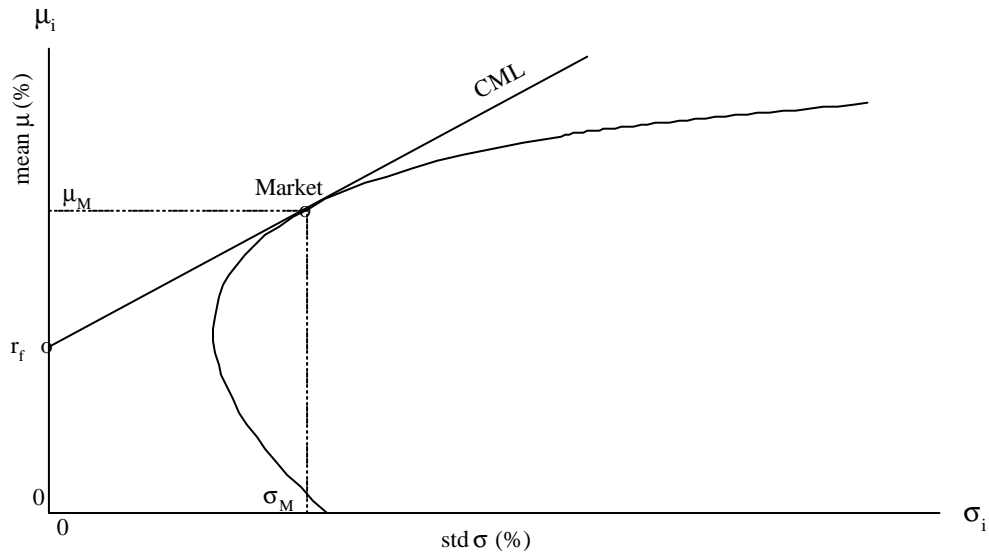


Figure 1: Equilibrium market portfolio and efficient frontier.

When we aggregate the portfolios of all individual investors, lending and borrowing will cancel out, and the value of the aggregated risky portfolio will equate the entire wealth of the economy. The tangency portfolio has become the equilibrium market portfolio.

# The Market Price of Risk

There are  $N$  mean-variance investors in the economy, each with \$1.

Investor	Risk Aversion	Portfolio Weight
1	$A_1$	$\frac{\mu_M - r_f}{\sigma_M^2 A_1}$
2	$A_1$	$\frac{\mu_M - r_f}{\sigma_M^2 A_2}$
3	$A_1$	$\frac{\mu_M - r_f}{\sigma_M^2 A_3}$
...	...	...
$N$	$A_N$	$\frac{\mu_M - r_f}{\sigma_M^2 A_N}$

Aggregating over all Investors, the total wealth invested in the market portfolio is:

$$\$1 \cdot \frac{\mu_M - r_f}{\sigma_M} \left[ \frac{1}{A_1} + \frac{1}{A_2} + \dots + \frac{1}{A_N} \right] \quad (1)$$

In equilibrium, the total wealth invested in the market portfolio must be:

$$\$1 \cdot N \quad (2)$$

This implies:

$$\mu_M - r_f = \sigma_M^2 \bar{A} \quad (3)$$

where  $\bar{A}$  is an overall measure of the risk aversion among the market participants:

$$\frac{1}{\bar{A}} = \frac{1}{N} \cdot \left[ \frac{1}{A_1} + \frac{1}{A_2} + \dots + \frac{1}{A_N} \right] \quad (4)$$

# Pricing the Individual Risky Assets

The market portfolio is made of individual risky assets:

$$r_M = w_1 r_1 + w_2 r_2 + \dots + w_N r_N \quad (5)$$

where  $w_i$  is the fraction of total market wealth invested in asset  $i$ .

How are the individual risky assets priced in equilibrium?

To answer this question, we deviate the equilibrium holding  $w_i$  of asset  $i$  slightly away from its optimal level, and see how such a deviation affect our investor's maximized utility.

Let's focus on a representative investor with the average risk aversion .

$$U(r) = E(r) - \frac{1}{2} \bar{A} \text{var}(r) \quad (6)$$

His portfolio holding:

$$r = y^* (w_1^* r_1 + w_2^* r_2 + \dots + w_N^* r_N) + (1 - y^*) r_f \quad (7)$$

What is his  $y^*$  ?

In equilibrium,  $w_i^*$  is the optimal solution for this investor. This means:

$$\frac{\partial U(r)}{\partial w_i} = 0 \quad (8)$$

Keeping everything else fixed, what if we change  $w_i$  a little?

$$\frac{\partial E(r)}{\partial w_i} = E(r_i) - r_f \quad (9)$$

$$\frac{\partial \text{var}(r)}{\partial w_i} = 2 \cdot \text{cov}(r_M, r_i) \quad (10)$$

So it must be that:

$$E(r_i) - r_f = \bar{A} \text{cov}(r_M, r_i) \quad (11)$$

Recall that:

$$E(r_M) - r_f = \bar{A} \text{var}(r_M) \quad (12)$$

This takes us to:

$$E(r_i) - r_f = \beta_i (E(r_M) - r_f) \quad (13)$$

where

$$\beta_i = \frac{\text{cov}(r_M, r_i)}{\text{var}(r_M)} \quad (14)$$

Our derivation uses the quadratic utility function. In general, the proof goes through for any utility function with a preference for mean, and aversion to variance.



# Risk and Reward in the CAPM

For a risky asset  $r_i$ , the right measure of the rewardable risk is not its variance  $\text{var}(r_i)$ , but its covariance  $\text{cov}(r_i, r_M)$  with the market.

The exposure to the market risk can be best quantified by:

$$\beta_i = \frac{\text{cov}(r_M, r_i)}{\text{var}(r_M)} \quad (15)$$

For one unit exposure to the market risk, the reward is the same as the market:

$$E(r_M) - r_f \quad (16)$$

For  $\beta$  unit of exposure to the market, the reward is:

$$\beta_i \cdot (E(r_M) - r_f) \quad (17)$$

For zero exposure to the market risk, the reward is zero, no matter how risky the asset is. In summary, the risk and reward relation in the CAPM is a linear relation.

## Systematic vs. Idiosyncratic

Each investment carries two distinct risks:

- *Systematic risk* is market-wide and pervasively influences virtually all security prices.

Examples are interest rates and the business cycle.

- *Idiosyncratic risk* involves unexpected events peculiar to a single security or a limited number of securities.

Examples are the loss of a key contract or a change in government policy toward a specific industry.

# The Security Market Line

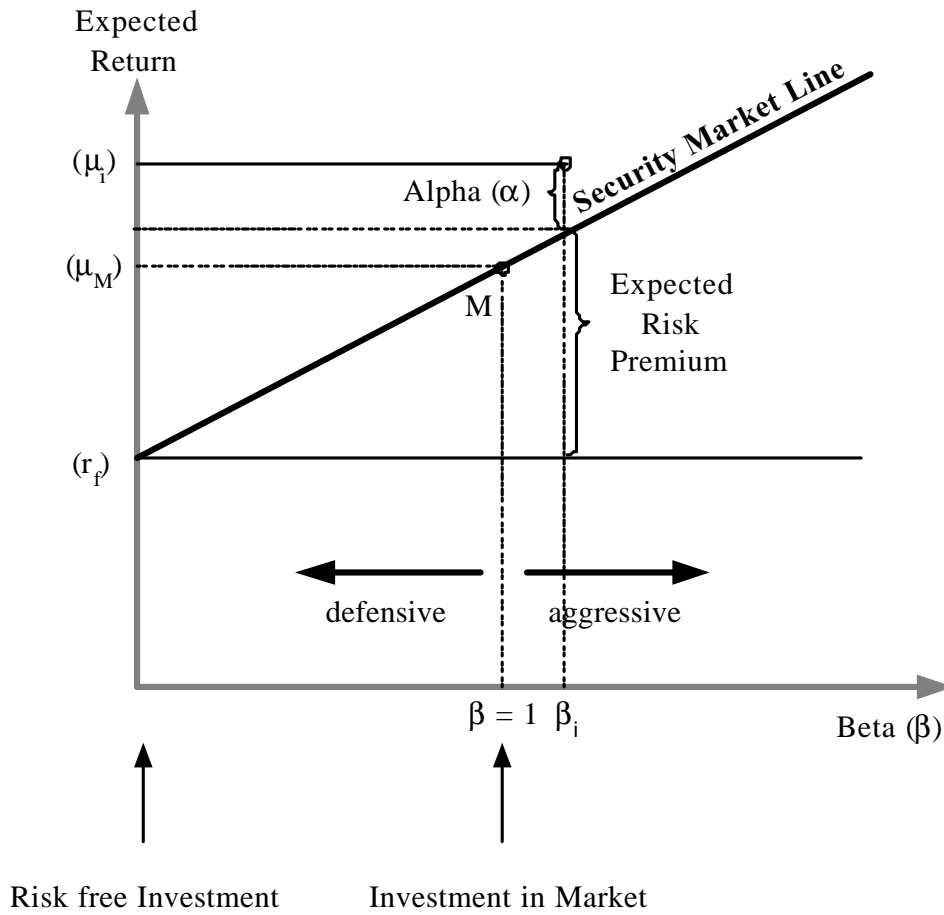


Figure 2: Equilibrium market portfolio and efficient frontier.

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# A Linear Factor Model

A simple model to capture these two types of risks is the linear factor model:

$$r_i = E(r_i) + \beta_i \cdot F + \varepsilon_i \quad (18)$$

which carries two risky components:

- *systematic*  $F$ : common to all securities.
- *idiosyncratic*  $\varepsilon_i$ : specific only to security  $i$ .

Both the common factor  $F$  and the idiosyncratic component are zero mean random variables.

# The Arbitrage Pricing Theory

## A SINGLE-FACTOR VERSION

Assume a frictionless market with no taxes or transaction costs. Assets are perfectly divisible. There is no short-sale constraint.

Assume a one-factor linear model:

- $\beta_i$  asset  $i$ 's sensitivity to the common factor.
- $F$  common factor, with  $E(F) = 0$ .
- $\varepsilon_i$  firm-specific return, with zero mean, and independent of the common factor or other firms' idiosyncratic component.

Possible common factor: unexpected changes in inflation, industrial production, etc.

The Arbitrage Pricing Theory:

$$\frac{E(r_i - r_f)}{\beta_i} = \frac{E(r_j - r_f)}{\beta_j} \quad (19)$$

The expected return of any asset is determined by its exposure to the common factor, and has nothing to do with its idiosyncratic component.

In deriving APT, one need not make any assumption about investors' preferences, or assume any specific distribution for the asset returns.

The APT is not an equilibrium concept. It does not rely on the existence of a market portfolio. It is based purely on no-arbitrage conditions.

# Summary

In equilibrium, the tangency portfolio becomes the market portfolio. The expected return of the market portfolio depends on the average risk aversion in the market.

The intuition of the CAPM: expected return of any risky asset depends linearly on its exposure to the market risk, measured by  $\beta$ .

Diversification is an important concept in finance. It builds on a powerful mathematical machine called Strong Law of Large Number.

Like the CAPM, the basic concept of the APT is that differences in expected return must be driven by differences in non-diversifiable risk.

The APT is based purely on no-arbitrage condition. It is not an equilibrium concept, and does not depend on having a market portfolio.

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**Focus:**

BKM Chapters 9-11

- p. 263 bottom to 284, (CAPM, assumptions, beta, liquidity, covariance, expectations, SML, zero-beta model, alpha)
- p. 287, eq. 10.5, eq.10.6 & eq.10.7
- p. 300 to 308
- p. 308 to 313 middle (eq. 10.15, eq. 10.16)
- p. 324 to 334 middle (diversification, eq. 11.2, APT and CAPM, Multifactor APT, eq. 11.5, eq. 11.6)

Reader: Roll and Ross (1995).

type of potential questions: chapter 9 concept check question 1, 2, 3 & 4, p. 286 ff. questions 1, 4, 17, 22, 23, 25 chapter 10 concept check question 1, 2, 3 & 4, p. 314 ff. questions 4, 5, 6, 7, 18, 19 chapter 11 concept check question 2, 3, 4 & 5, p. 335 ff. questions 3, 5, 7, 10, 16

# Questions for Next Class

Please read:

- BKM Chapter 13,
- Jagannathan and McGrattan (1995),
- Kritzman (1993), and
- Kritzman (1994)

Think about the following questions:

- What are the predictions of the CAPM?
- Are they testable?
- What is a regression?
- What is t-test?

# TECHNICAL NOTES: The Mathematical Foundation of Diversification

Behind the concept of diversification, there is deep, elegant, and powerful mathematical machinery.

STRONG LAW OF LARGE NUMBERS: Let  $x_1, x_2, \dots$ , be a sequence of identically distributed and independent random variables with mean  $\mu$ . We have:

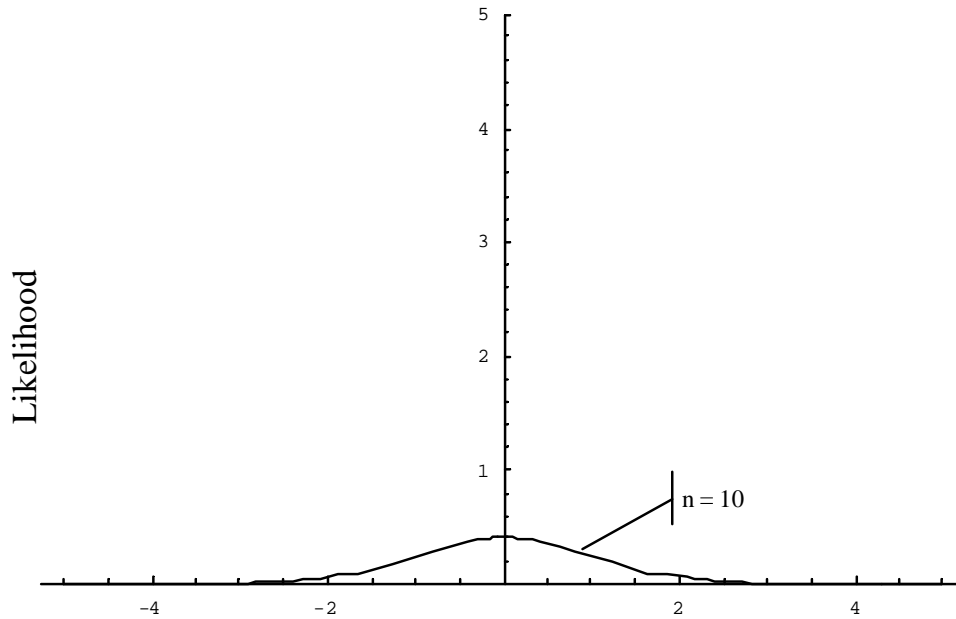
$$\lim_{N \rightarrow \infty} \frac{1}{N} x_i = \mu \tag{20}$$

almost surely.



# A Simulation Based "Proof"

STEP 1: We start with  $X_1$ , assume that it is standard normal, and simulate it 10,000 times. Using the 10,000 scenarios, we plot its "empirical" probability distribution, which is basically the histogram of the 10,000 scenarios normalized so that the total probability is one. For notational purpose, we write  $y_1 = x_i$ .



*Fig-*

*ure 3: Likelihood and simulation outcome for  $y_1$  at  $n=1$ .*

STEP 2: We repeat STEP 1 ten times, each time with a new seed in our random number generator, so that  $x_1, x_2, \dots, x_{10}$  are indeed independent. We add them up, scenario by scenario, and re scale the sum by 10. That is, we have 10,000 scenarios of

$$y_{10} = \frac{1}{10} (x_1 + x_2 + \dots + x_{10}) \quad (21)$$

We then plot the "empirical" probability distribution of  $y_{10}$ . Chart:

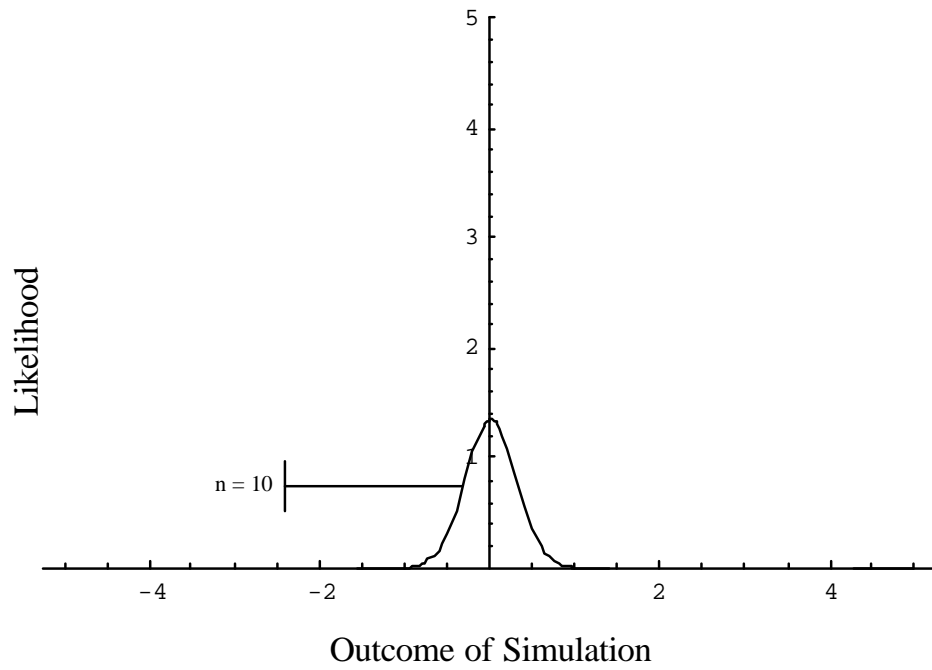


Figure 4: Likelihood and simulation outcome for  $y_1$  at  $n=10$ .

STEP 3: Finally, we repeat STEP 1 one hundred times, obtaining 10,000 scenarios of:

$$y_{100} = \frac{1}{100} (x_1 + x_2 + \cdots + x_{100}) \quad (22)$$

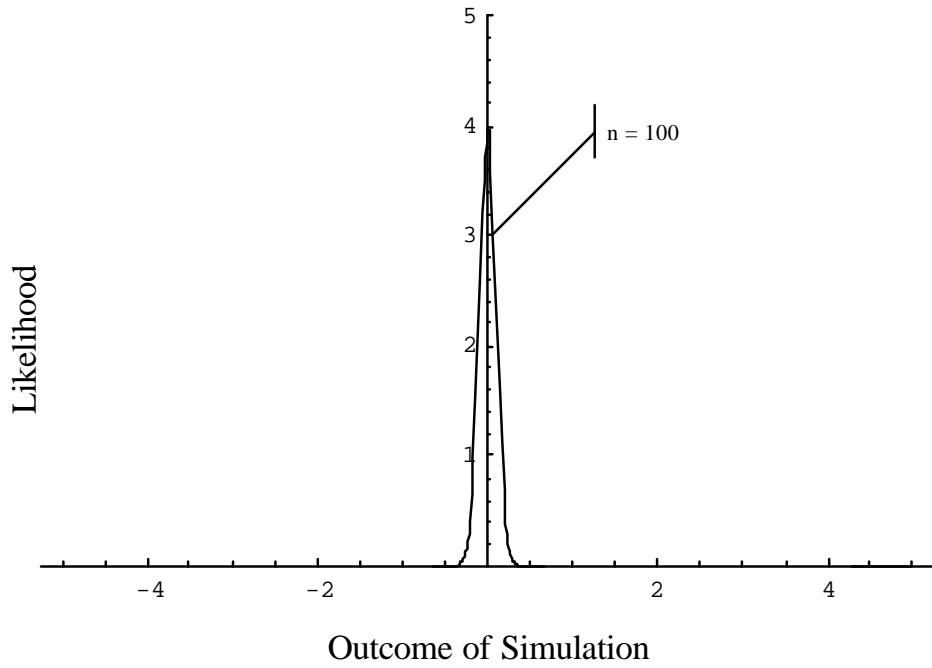


Figure 5: Likelihood and simulation outcome for  $y_1$  at  $n=100$ .

According to Law of Large Number, as  $N$  grows infinitely large,

$$y_N = \frac{1}{N} \sum_{i=1}^N x_i \quad (23)$$

approaches zero with probability one.

For  $x_i$  normally distributed, it is actually easy to see that  $y_N = \frac{1}{N} \sum_{i=1}^N x_i$  will approach zero with probability one. Since:

$$\text{var}(y_N) = \frac{1}{N^2} \sum_{i=1}^N \text{var}(x_i) = \frac{1}{N} \quad (24)$$

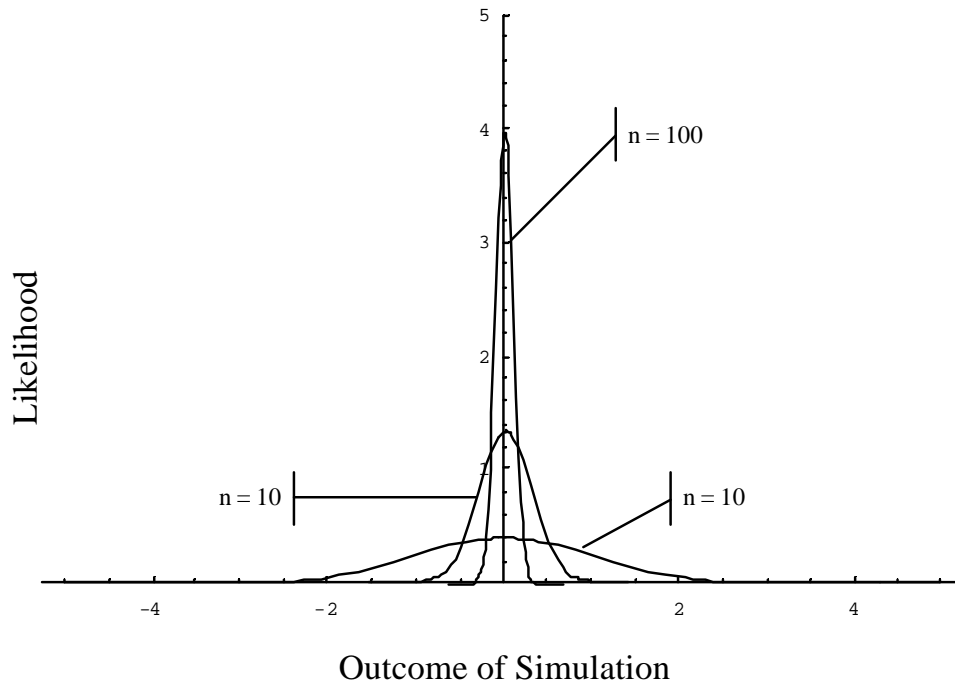


Figure 6: Likelihood and simulation outcome for  $y_1$  at  $n=1 / 10 / 100$ .