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Chapter 6: Measuring Risk–Dynamic Models

Part A—The Random Walk Model of Stock Prices

Recent decades have witnessed revolutionary advances in the tools used to model risk and to price risky financial assets. The Black-Scholes-Merton approach to option pricing has been generalized to all corporate liabilities, including stocks, bonds and derivatives of all types. This approach is now commonly used to price mortgage obligations, currency agreements, insurance contracts and myriad other financial instruments. Most recently, the approach has been extended to the valuation of real options, investments in real assets such as the development of oil fields, the operation of electric generating plants and the negotiation of options to purchase aircraft.

The engine under the hood of every real options model is a stochastic process model of one or more critical variables or factors—the oil price, for example. A stochastic process model is a parsimonious description of the dynamics governing the evolution of the variable through time. It captures long-term trends—such as any forecasted long-term increase in the price. It captures short-term dynamics such as seasonal fluctuations, as well as the tendency of sudden price shocks to reverse themselves. And it captures all elements of uncertainty, both short-term shocks that are dissipated and long-term shocks that persist and are compounded. Once the stochastic process model is specified and its parameters estimated, an analyst can value all varieties of real assets with cash flows tied in any complicated way to the factor.

How well the engine runs depends upon what you put in it. If the model accurately captures the dynamics—both in general structure, as well as in the precise values of the parameters—then the risk assessment and valuations made using the model will be useful. If the model is inaccurate, then, …well, then we have a problem. So it is critical for the corporate manager to understand the essential characteristics of alternative models and be able to evaluate the fidelity of the model to the dynamics of the factors and risks driving their business.

This chapter is predicated on a distinction between the underlying factor and the derivative asset. First we model the underlying factor. Then we determine how the cash flows or value of an asset is derived from this underlying factor – hence the term ‘derivative’. Then we try to understand how the risk from the underlying factor is translated into risk for the asset.

Some students will be familiar with the tools here as developed for modeling stocks and other financial instruments. Our analysis is meant to apply more generally. Any variable can conceivably be a factor, and we need to think about the many ways that risk should be modeled for the full diversity of variables that can be factors. Because our analysis is meant to be very general, students should be cautious about applying all of the results learned in the context of modeling financial assets. The equilibrium conditions determining the prices of financial assets impose specific conditions on the models used. Since we are modeling general variables, not just financial assets, the equilibrium conditions may not apply and the models may have properties different from what the student is familiar with.

In this chapter we have tried to strike a difficult balance. We present the bare essentials of stochastic process modeling, attempting to minimize the mathematical detail, while still providing enough material to enable the reader to actually implement and evaluate these models. We present a broad selection of models which embody very different structures of risk and which we believe give the reader a good feel for the various risk patterns that can be modeled. But, of course, the full library of potential models is much larger, and the true specialist will know that we have only touched the surface. We have tried to select models that we believe provide a useful representation of relevant factors—for example, a good model of the oil price that can be reliably used to assess the risk of major oil related capital investments. In doing so, we have chosen to brush aside aspects that we believe are of lesser significance for corporate managers, although they may be significant for short-term money managers speculating on oil futures. All of these choices involve judgment and reasonable people may differ. This is our best shot.

6.1 Discrete Time Stochastic Process

We start with the standard discrete time stochastic process model for stock prices, the random walk model. Suppose that we want to analyze a stock with an average annual return of μ=12%, also known as the drift, and volatility σ=22%. Suppose further that we are analyzing the stock’s possible movement through the horizon T=2. We divide this into N total periods or n periods per year, $n=N/T$, where the length of each period is $\Delta t=1/n$. If n=12, then each period is a month and Δt=0.0833, if n=52, then each period is a week and Δt=0.0192, if n=250, then each period is a trading day and Δt=0.0040, and so on. We mark off the N points in time as t_0, t_1 … t_i,… t_N, with t_0=0 and t_N=T. Figure 6.1 below illustrates this structure.

The initial stock price, $S(t_0)$, is given. The return earned over the period from t_0 to t_1, $R(t_1)$, and therefore the next period stock price, $S(t_1)$, are random variables. We define the return as the continuously compounded return so that:

$$S(t_i) = \quad S(t_{i-1})e^{R(t_i)}, \tag{6.1}$$

or equivalently,

$$R(t_i) = \quad \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right) = \ln\big(S(t_i)\big) - \ln\big(S(t_{i-1})\big). \tag{6.2}$$

An important property of continuously compounded returns is that they are additive: the cumulative return over any horizon, $R(t_0,t_i)$, is equal to the sum of the returns over that horizon. One can see this by analyzing the recursive calculation of the price:

$$S(t_1) = \quad S(t_0)e^{R(t_1)}$$

and

$$S(t_2) = \quad S(t_1)e^{R(t_2)},$$

therefore

$$S(t_2) = \quad S(t_0)e^{R(t_1)}\,e^{R(t_2)},$$

or, equivalently

$$S(t_2) = \quad S(t_0)e^{R(t_1)+R(t_2)},$$

which, by definition, is the cumulative return,

$$S(t_2) = \quad S(t_0)e^{R(t_1,t_2)}.$$

This recursive substitution can be repeated any number of periods. Therefore, we are free to write

$$R(t_0,t_i) = \quad \sum_{k=1}^{i} R(t_k) \tag{6.3}$$

and,

$$S(t_i) = \quad S(t_0)\, e^{R(t_0,t_i)} \,. \tag{6.4}$$

We assume each return is an independently and identically distributed random variable from a normal distribution with mean m and variance v^2:

$$R(t_i) \quad = m + v\,\tilde{\varepsilon}_i \tag{6.5}$$

where ε_i is a standard normal random variable. The mean and variance of the period return depend on the size of the period, Δt. In order that the annual return and variance equal μ and σ as assumed above, we set m and v as follows:

$$m(\Delta t) = \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t\,, \tag{6.6}$$

$$v(\Delta t) = \sigma\sqrt{\Delta t}\,. \tag{6.7}$$

The rationale for this definition will become clear shortly.

6.2 Monte Carlo Simulation

We can easily simulate this stochastic process. First we need a series of N draws of the standard normal random variables, ε_1, ε_2, … ε_N, which can be readily generated in a standard Excel spreadsheet or with any of a number of other mathematical programs. We then calculate the N return variables $R(t_1)$, $R(t_2)$, … $R(t_N)$ using equation (6.5), and then calculate the N cumulative return variables, $R(t_0,t_1)$, $R(t_0,t_2)$, … $R(t_0,t_N)$ using equation (6.3) and the N stock price variables, $S(t_1)$, $S(t_2)$, … $S(t_N)$ using equation (6.4). This gives us a sample path for the stock price.

Table 6.1 shows a sample of the first 10 draws of the standard normal random variable and the calculation of returns, cumulative returns and stock prices assuming input parameters μ=12% and σ=22% and periods of a week's length, n=52 and Δt=0.0192. For these parameters we have m=0.18% and v=3.05%.

Figure 6.2 shows how a sample path of the stock price series might appear given three different choices for the length of a period—one month, one week and one trading day.

Figure 6.3 shows 4 different sample paths of the stock price generated by the same parameters, but using different sets of draws of the standard normal random variables, ε_1, ε_2, … ε_N. Also shown in Figure 6.3 are the corresponding sample paths of cumulative returns.

In constructing our simulation, we used the intermediate step of calculating cumulative returns and then stock prices, when it would have been possible to move directly from returns to stock prices by applying equation (6.1) recursively. There are a number of reasons why it is useful to construct our simulation and perform much of the analysis in cumulative returns instead of directly in terms of stock prices. Once we have constructed the cumulative return at any horizon, it is a simple matter to translate that back into a stock price. One reason for working in returns is that the scale is invariant. In a graph of cumulative returns, a five percentage point change looks the same, regardless of whether it is a five percentage point change from a low cumulative return or from a high cumulative return. In a graph of stock prices, a five percentage point change looks smaller if it is a change from a low stock price and looks larger if it is a change from a high stock price. Since stock prices tends to grow exponentially, this tends to exaggerate the significance of later returns as compared to earlier returns and distorts our ability to grasp the dynamics. Working in returns is also useful in minimizing the impact of the rounding errors that creep into calculations with many periods. The cumulative impact of these errors is smaller if we first sum returns and only exponentiate the cumulative return at the conclusion of the analysis in order to translate the results back into stock prices.

By generating repeated paths of the series of returns and stock prices we can produce a histogram of values for the stock price and the cumulative return at any point in time, $t \in \{t_1 \ldots t_i, \ldots t_N\}$. Figure 6.4 shows a histogram of cumulative returns for a simulation of 100 sample paths over a horizon of T=2 years using N=100 periods in total or n=50 per year, with μ=12% and σ=22%.

For a large enough set of paths, this histogram should approximate the true probability distribution for the process we are simulating, and so can be used to estimate an answer for certain standard types of probability questions. What is the expected cumulative return to T=2? In this small sample, the mean cumulative return at T=2 is 21.2%. What is the standard deviation of cumulative returns at T=2? The sample standard deviation is 28.1%. What is the probability that the stock price at 2 years is greater than \$90? In our small sample, 86% of the paths end with a stock price greater than \$90 at T=2. What is the expected stock price at T=2, given that it is greater than \$90? In our sample, the average stock price at T=2 among those paths for which the price is greater than \$90, is \$136.62. What is the probability that between t=0 and t=T the price is always above \$90 within the 2 year horizon? In our sample, 46% of the paths are always above

$90 throughout the entire window. As tedious as these types of calculations are, they are nevertheless readily doable with a computer.

Obviously, the accuracy of our estimated answers to these questions depends upon the size of the sample we take. A sample of 100 is useful for getting an initial feel for a problem, but is far too small for reliable results on any interesting questions. It is common to see results presented using a sample size of 10,000 runs, but there is nothing sacrosanct about this number. The right sample needed depends upon the degree of accuracy required and the particular function being estimated. The accuracy also depends on other elements of the simulation. For example, the formula and procedure used to generate the random number can affect the accuracy. Also, it should be clear that simply reproducing the distribution in the way we have described—simple sampling—is a sort of brute force technique. A number of techniques have been developed to deliberately select a sample that most efficiently reflects the properties of the underlying distribution, i.e., using the smallest sample size. See, for example, Latin hypercube sampling or orthogonal sampling. These techniques will not be explored in any more detail here.

More important than the size or technique of sampling, of course, is the question of whether the mathematical model we are using is the right one and whether the parameter values we have selected are right. As always, we are subject to the dictum ‘garbage in, garbage out.’

6.3 The Normal Distribution of Returns

Earlier we assumed that each period’s return was normally distributed and that each period’s return was independently and identically distributed. This will give us a simple expression for the probability distribution of returns at every horizon, which in turn will enable us to derive explicit formulas for the types of questions we had asked earlier, questions about the likely and conditional values of the stock price and returns.

The sum of a set of normal random variables is itself normally distributed. Because the returns each period are independently distributed, the mean of the sum is the sum of the means. And because they are identically distributed, the sum of the mean returns is simply proportional to the number of periods, i, or elapsed time, t_i:

$$E[R(t_0,t_i)] = E\left[\sum_{k=1}^{i} R(t_k)\right] = \sum_{k=1}^{i} E[R(t_k)] = \sum_{k=1}^{i} m = mi\,.$$

Noting that the mean return in a single period, *m*, is actually a function of the length of the period, $m(\Delta t)$, we can rewrite this as

$$E\left[R\left(t_0,t_i\right)\right]=m(\Delta t)\,i=\left(\mu-\tfrac{1}{2}\sigma^2\right)\Delta t\,i=\left(\mu-\tfrac{1}{2}\sigma^2\right)t_i\,. \tag{6.8}$$

The assumption of independently distributed returns also implies that the variance of the sum of returns is equal to the sum of the variances of the returns. And since the returns are identically distributed, the variance of the sum of returns is also linear in the number of periods, *i*, or elapsed time, t_i:

$$Var\left[R\left(t_0,t_i\right)\right]=Var\left[\sum_{k=1}^{i}R\left(t_k\right)\right]=\sum_{k=1}^{i}Var\left[R\left(t_k\right)\right]=\sum_{k=1}^{i}v^2=v^2\,i\,.$$

Noting that the standard deviation of the return in a single period, *v*, is actually a function of the length of the period, $v(\Delta t)$, we can rewrite this as

$$Var\left[R\left(t_0,t_i\right)\right]=v(\Delta t)^2\,i=\sigma^2\,\Delta t\,i=\sigma^2\,t_i\,. \tag{6.9}$$

Therefore, the cumulative return from period 0 to period *i* is normally distributed, with mean *mi* and variance vi^2:

$$R(t_0,t_i)\sim N(mi,v^2i),$$

or, equivalently, the cumulative return from period 0 to period *i* is normally distributed, with mean $(\mu-½\sigma^2)t$ and variance σ^2t:

$$R\left(t_0,t_i\right)\sim N\left(\left(\mu-\tfrac{1}{2}\sigma^2\right)t_i,\sigma^2t_i\right).$$

The top panel of Figure 6.5 shows the probability distribution of cumulative returns over a horizon of T=2 years with μ=12% and σ=22%. It should be compared against the histogram of returns shown in Figure 6.4. The mean of the distribution is a return of 19.2%. This should be compared against the mean in the sample which was 21.2%. The difference between those two values reflects the error we get because our small sample turns out not to be wholly representative of the full distribution. The larger the sample size the smaller this error is likely to be. The standard deviation of the distribution shown in Figure 6.5 is 31.1%. This should be compared against the standard deviation of the sample which was 28.1%.

The top panel of Figure 6.6 shows the expected cumulative return at each horizon using equation (6.8). Also shown are the one standard deviation confidence bounds which correspond to the 68% confidence interval using these equations:

$$UR(t_0,t_i) = (\mu - \tfrac{1}{2}\sigma^2)t_i + \sigma\sqrt{t_i} \tag{6.10}$$

$$LR(t_0,t_i) = (\mu - \tfrac{1}{2}\sigma^2)t_i - \sigma\sqrt{t_i}\ . \tag{6.11}$$

One can see in the figure as well as in equation (6.8) that the expected cumulative return grows linearly in time. One can see in equation (6.9) that the variance of return also grows linearly in time. This means that the standard deviation or volatility of return grows as the square root of time, and one sees this in the shape of the confidence bounds graphed in Figure 6.6 and in the form of equations (6.10) and (6.11).

6.4 The Lognormal Distribution of Prices

The expected value and variance of the stock price variable is slightly more complicated since the stock price is equal to the exponentiated return. Rewriting the relationship between the stock price and returns shown in equation (6.2) we have

$$\begin{aligned} \ln\big(S(t_i)\big) &= \ln\big(S(t_{i-1})\big) + R\big(t_i\big) \\ &= \ln\big(S(t_{i-1})\big) + m + v\tilde{\varepsilon}_i\ . \end{aligned} \tag{6.12}$$

Since the return is a normally distributed random variable, equation (6.12) implies that the log of the price is normally distributed. In that case, the price itself is lognormally distributed:

$$S(t_i) \sim \text{Log-N}\big(\ln(S(t_{i-1}) + m, v^2\big),$$

or, alternatively,

$$S(t_i) \sim \text{Log-N}\big(\ln(S(t_0) + (\mu - \tfrac{1}{2}\sigma^2)t_i, \sigma^2 t_i\big).$$

The bottom panel of Figure 6.5 shows a lognormal distribution. Contrast the normal distribution of cumulative returns shown in the top panel of Figure 6.5 with the lognormal distribution of prices. A lognormally distributed random variable can never go below zero, which is an appropriate feature for a distribution describing stock prices. But this contributes to the fact that the lognormal is not a symmetric distribution. It is skewed to the left, with a long upper tail. Consequently, the median of the distribution will lie to the left of the mean, which is an important property in understanding the relationship between expected returns and expected prices. The mean return in the top panel is also the median return. Looking back at the bottom panel of Figure 6.3, the histogram of stock prices from our Monte Carlo simulation, you can see the features of a lognormal distribution there, too, and you can contrast these with the features of the normal distribution in the histogram of stock returns immediately above it.

Standing at t_0, the expected value of the stock price at t_1, $S(t_1)$, is given by:

$$E\big(S(t_1)\big) = S(t_0)\, e^{m+\frac{1}{2}v^2}.$$

Note that the volatility parameter, v, enters into the expectation, increasing the expected value. Consequently, the expected stock price is greater than the price corresponding to the expected return:

$$E\big(S(t_1)\big) = S(t_0)\, e^{m+\frac{1}{2}v^2} \quad > \quad S(t_0)\, e^{m} = S(t_0)\, e^{E(R(t_1))}.$$

This is because the variance component of the return makes its own contribution to the expected stock price:

$$E\big(S(t_1)\big) = E\Big(S(t_0)\, e^{R(t_1)}\Big) = E\Big(S(t_0)\, e^{m+v\tilde{\varepsilon}_1}\Big) = S(t_0)\, e^{m}\; E\Big(e^{v\tilde{\varepsilon}_i}\Big).$$

Although the mean of the random term ε_1 is zero, the expected value of the exponentiated random term is not zero:

$$E\Big(e^{v\tilde{\varepsilon}_1}\Big) = e^{\frac{1}{2}v^2}.$$

The expected stock price through time is therefore given by

$$E\big(S(t_i)\big) = S(t_0)\, e^{\left(m+\frac{1}{2}v^2\right)i}.$$

Noting that the mean return in a single period, m, is actually a function of the length of the period, $m(\Delta t)$, we can rewrite this as

$$E\big(S(t_i)\big) = S(t_0)\, e^{\left(\mu-\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2\right)t_i} = S(t_0)\, e^{\mu t_i}. \tag{6.13}$$

Our earlier choice of parameterization for the per period mean return, writing $m(\Delta t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t$, allowed us here, in calculating the mean price, to cancel the terms where the volatility enters and obtain an expression exclusively in terms of the single parameter, μ.

The bottom panel of Figure 6.6 graphs the expected stock price through time from equation (6.13). Also graphed is the median stock price, which is the price corresponding to the median return given in equation (6.8):

$$S(t_0)\, e^{\left(\mu-\frac{1}{2}\sigma^2\right)t_i}. \tag{6.14}$$

The median stock price is always less than the mean stock price. Finally, the bottom panel of Figure 6.5 also graphs the upper and lower bounds on the 68% forecast confidence interval for the stock price. These are calculated by exponentiating the upper and lower bounds confidence interval for returns in equations (6.10) and (6.11):

$$US(t_i) \quad = S(t_0)\, e^{UR(t_0,t_i)}, \text{ and} \tag{6.15}$$

$$LS(t_i) \quad = S(t_0)\, e^{LR(t_0,t_i)}. \tag{6.16}$$

6.5 Probability Calculations

Earlier, we used Monte Carlo simulation to estimate answers to questions such as, what is the probability that the stock price at T=2 is greater than \$90? Monte Carlo simulation is a very empirical way to approach this question, but one that requires the brute force of many calculations to implement. Because we now have a convenient characterization of the distribution of prices at every horizon, we can directly derive the precise answer to this question without going through the full simulation process. We start by noting that $S(t_i)$>X exactly when ln($S(t_i)$)>ln(X), so that:

$$\Pr\left(S(t_i) > X\right) = \Pr\left(\ln\left(S(t_i)\right) > \ln(X)\right)$$

Since ln($S(t_i)$ is normally distributed, if we subtract the mean and divide by the standard deviation, we will transform it to a standard normal random variable for which the relevant probabilities are readily to hand. Doing this to both sides of the expression inside the probability function gives us:

$$= \Pr\left(\frac{\ln\left(S(t_i)\right)-\ln\left(S(t_0)\right)-\left(\mu-\frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}} > \frac{\ln(X)-\ln\left(S(t_0)\right)-\left(\mu-\frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}}\right)$$

$$= \Pr\left(z > \frac{\ln(X)-\ln\left(S(t_0)\right)-\left(\mu-\frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}}\right),$$

where z is a standard normally distributed random variable. Taking advantage of symmetry around zero in the standard normal distribution, we can rewrite this as

$$= \Pr\left(z < -\frac{\ln(X) - \ln(S(t_0)) - \left(\mu - \frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}} \right)$$

$$= \Pr\left(z < \frac{\ln(S(t_0)) - \ln(X) + \left(\mu - \frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}} \right)$$

$$= N\left(\frac{\ln(S(t_0)) - \ln(X) + \left(\mu - \frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}} \right),$$

where $N(\ )$ is the cumulative normal distribution function. The expression inside the parenthesis is of a form that appears in the Black-Scholes equation for the price of a call option. Black-Scholes used the variable d_2 for this expression. Since our expression is similar, but slightly different, we will use the variable $\hat{d}_2$:

$$\hat{d}_2 = \frac{\ln(S(t_0)) - \ln(X) + \left(\mu - \frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}}, \tag{6.17}$$

so that we have

$$\Pr\left(S(t_i) > X\right) = N\left(\hat{d}_2\right). \tag{6.18}$$

We can solve for the complementary probability that the price ends up below the value X,

$$\Pr\left(S(t_i) < X\right) = N\left(-\hat{d}_2\right). \tag{6.19}$$

We will come back to analyze the expression for $\hat{d}_2$ in more detail when we come to pricing risk and the valuation of a call option. We can evaluate equations 6.18 and 6.19 in Excel using the NormSDist function. For our assumed parameters of T=2 years, μ=12% and σ=22%, we have $\hat{d}_2$=0.95447 and we arrive at the solution that Pr($S(t_i)$>X)= 83%. This should be contrasted with the value of 86% from our Monte Carlo simulation.

Earlier we had also asked, what is the expected stock price at T=2 given that it is greater than $90? This, too, can be given a precise answer without resort to simulation.

$$E\left[S(t_i)\,\middle|\,S(t_i) > X\right] = \int_X^{\infty} S(t_i)\; f\left(S(t_i)\right) dS(t_i) \Bigg/ \int_X^{\infty} f\left(S(t_i)\right) dS(t_i)$$

$$= S(t_0)\; e^{\mu t_i}\; N\left(\frac{\ln\left(S(t_0)\right) - \ln\left(X\right) + \left(\mu + \frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}}\right) \Bigg/ N\left(\hat{d}_2\right)$$

$$= S(t_0)\; e^{\mu t_i}\; N\left(\hat{d}_1\right) \Big/ N\left(\hat{d}_2\right) \qquad (6.20)$$

where,

$$\hat{d}_1 = \frac{\ln\left(S(t_0)\right) - \ln\left(X\right) + \left(\mu + \frac{1}{2}\sigma^2\right)t_i}{\sigma\sqrt{t_i}} = \hat{d}_2 + \sigma\sqrt{t_i}\;. \qquad (6.21)$$

The variable $\hat{d}_1$ is also comparable to the variable d_1 which appears in the Black-Scholes equation. More on that later. Using equations 6.20 and 6.21, in our numerical example, we have $\hat{d}_1$=1.26559 and the expected stock price at T=2 given that it is greater than \$90 is \$137.40.

6.6 Estimating the Parameters

Until now, we have assumed a given stochastic process driving the evolution of the stock price, and we have assumed values for the pair of parameters defining that process, μ and σ. Then we have generated probabilistic forecasts of what realization of stock returns and prices we would be likely to see. In reality, neither the process nor the parameters are given. We observe a history of past stock returns and prices, and we infer the structure of the process, including the values for the parameters μ and σ.

For the moment, we are going to continue with the assumption that we know the general structure of the stochastic process driving returns and prices. However, we are going to acknowledge that we do not necessarily know the values for the parameters μ and σ. How do we estimate the parameters μ and σ from a history of stock price data?

One of the attractive things about the stochastic process model we have been using is that the estimation of the two parameters is very simple. Given a sample of N+1 stock price variables, $S(t_0)$, $S(t_1)$, $S(t_2)$, … $S(t_N)$ we first calculate the sequence of N return variables $R(t_1)$, $R(t_2)$, … $R(t_N)$, where $R(t_i)$=ln($S(t_i)$)/ln($S(t_{i-1})$). Now we recall that the per period expected return is m, and the volatility in per period returns is *v*. Estimates for *m* and *v* are simply the sample mean return and sample standard deviation of returns:

$$\hat{m} = Mean = \overline{R} = \sum_{i=1}^{N} R(t_i) \Big/ N \,, \tag{6.22}$$

$$\hat{v} = StDev = \sqrt{\sum_{i=1}^{N} \left(R(t_i) - \overline{R}\right)^2 \Big/ (N-1)} \,. \tag{6.23}$$

Recalling that these estimators have been calculated from returns calculated over periods of length $\Delta t = 1/n$, whereas the parameters μ and σ are denominated as annual values, we need to annualize the results by multiplying times the number of periods. In addition, we have to take care to adjust the mean return estimator with one-half the variance:

$$\hat{\mu} = (\hat{m} + \tfrac{1}{2}\hat{\sigma}^2)\, n \,, \tag{6.24}$$

$$\hat{\sigma} = \hat{v}\,\sqrt{n} \,. \tag{6.25}$$

Table 6.2 shows the estimation of μ and σ based on a short sample of observed stock prices. Because the sample is so short, both estimates have significant error. A larger number of observations is obviously required.

There are two ways to obtain a larger number of observations: (i) observe the price more frequently, increasing n and decreasing the length of a period, Δt, while keeping the horizon, T, constant, or (ii) extend the horizon, T, keeping the length of a period, Δt, constant. Increasing the frequently of observations, increasing n, and decreasing the length of a period, Δt, improves the precision of the estimate of volatility, but does not improve the precision of the estimate of drift. Extending the horizon improves the precision of the estimate of both drift and volatility, but it improves the precision of drift most. Nevertheless, a good estimate of the drift generally takes a very long horizon of data.

An intuitive way to understand why increasing the frequency of observations improves the precision of the estimate of volatility, but not of drift, and to understand why increasing the horizon is most useful for improving the estimate of drift is to examine the portion of a single period's return determined by each: Look at the ratio of the standard deviation of the return in a period, $v(\Delta t)$ to the mean return:

$$x = \frac{v(\Delta t)}{m(\Delta t)} = \frac{\sigma\sqrt{\Delta t}}{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t} = \frac{\sigma}{\left(\mu - \frac{1}{2}\sigma^2\right)\sqrt{\Delta t}}.$$

As the length of the period becomes very short, with Δt approaching zero, this ratio goes to infinity. This means that what we observe in a very, very short period reflects volatility. As the length of the period grows, with Δt approaching infinity, this ratio goes to zero, so that the majority of what we are observing reflects the drift. Unfortunately, it takes a long time to get to infinity, so it is difficult to get the information that we would like on the drift. It is much easier to make more frequent observations and improve our estimate of volatility. In theory, greater and greater frequency of observations will eventually give us a perfectly precise estimate of volatility. In practice, there are extra elements of noise in frequent observations – such as bid-ask bounce or a lack of trading, or simply mis-reporting of data – which put a limit on the value of the extra frequency of observations and a bound on how precise can our estimates of volatility become. We abstracted from these extra elements of noise when we structured the assumptions of our model.

6.7 The Continuous Time Representation

In the previous material, we modeled the stock price as a discrete time stochastic process. Each year was divided into n periods of length Δt. Suppose we increase the number of periods a year, letting the length of each period get smaller and smaller. If we continue this to the limit, so that Δt is infinitesimally small, then we have a continuous process. We can write the process two ways. The first is directly in terms of the stock price:

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dz, \tag{6.26}$$

The term $dS(t)$ is the continuous time equivalent of the per period change in price, $S(t_i)$-$S(t_{i-1})$. The term dt is the continuous time equivalent of Δt. The term dz is the continuous time equivalent of the standard normal random variables, ε_i. However, this simple statement hides the very complicated mathematical properties embedded in it. The term dz is called the Brownian motion. The key feature of the Brownian motion is that variation in the value of the process are normally distributed with a variance that is proportional to the length of the time period. Although the volatility parameter in equation (6.26) does not, on its face, appear to be multiplied times the square root of the time period—there is no dt multiplying the σ—this is only because the time element is implicit in the Brownian motion term, dz.

This second way to write this process is in terms of the log of the price, i.e., in terms of the returns:

$$d\ln\left(S(t)\right) = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma\, dz\,, \tag{6.27}$$

All of the properties we derived earlier for the discrete time process apply as well to this continuous time version. Returns are normally distributed, with the mean and variance of the return proportional to the horizon over which the return is compounded. Stock prices are lognormally distributed, and the expected stock price at t is $S_0\, e^{\mu}$. All of the earlier functions for the expected price and returns, confidence bounds and probability distributions remain valid.

Continuous time models are very mathematically demanding to develop and manipulate. But the investment often pays off since the mathematics enables development of powerful insights that can be difficult to grasp as readily when working in discrete time. Nevertheless, it is beyond the scope of this book to provide the reader with the tools necessary to work directly in continuous time. Instead, we simply exploit the continuous time results that have been produced and published in the literature, without attempting to derive them here. And we will try to make clear the correspondence between a given continuous time model and an alternative discrete time representation so that the reader develops the ability to perform as an intelligent consumer of future results in continuous time mathematics. In general, we will limit ourselves in this book to implementing solutions and simulations using discrete time processes, even where we regularly report on a particular continuous time model.

6.8 The Binomial and Other Tree or Lattice Representations

In this section we present a third way to model the random walk process, made popular by one well known binomial version. The general idea is to represent the evolution of the price through a series of branches in a tree. As with the discrete time stochastic process model, the model hypothesizes a series of discrete steps that can be specified as longer or shorter time intervals. However, at each point in time there are only a finite number of possible outcomes—in the binomial model, unsurprisingly, just two. With a large enough number of steps, the final distribution is very dense, despite the narrow range at each instant.

Using a tree structure has a couple of benefits. One is pedagogical. By reducing the number of possible outcomes at each time, it is possible to describe very simply the key value relationships from period to period. This makes the solutions transparent and easy to understand. Therefore the binomial model has become a popular teaching tool.

Second, using the tree structure expands the range of valuation problems we can tackle. Valuation is inherently forward looking and requires understanding the full structure of future contingencies. Current values are calculated by discounting future cash flows. In this sense, valuations starts from the end in time and moves backwards. To know today's value, we first need to know the distribution of possible values at the end of one year and then work backward. The Monte Carlo method works the other ways around. It starts from the current parameter value and generates a distribution for later dates. This works well for the single parameter being simulated, for which the governing stochastic process has been exogenously specified. But it doesn't work well for determining the value of assets with cash flows that are tied to that underlying parameter. With a binomial tree, once we know how the underlying parameter value evolves along the nodes of the tree, we can also work backwards and calculate the values of an asset with cash flows tied to that underlying parameter, and we can then learn how the value of that asset evolves through time.

Figure 6.7 shows the first step in a binomial tree. The tree begins at t=0 with a single node. The initial price is S_0=\$10.00. The log price is 2.303. The tree then has two branches leading to two nodes, the up node and the down node. The top branch represents the possibility of a high return, U=31.6%. The new log price is 2.618 and the new price is S_U=\$13.71. The bottom branch represents the possibility of a low return, -12.4%. The new log price is 2.178 and the new price is S_D=\$8.83.

Figure 6.8 shows how the tree branches out from the first step to the second step. From the top node at t=1, the tree again branches twice. The top branch once again represents the possibility of a high return, U=31.6%. The new log price is 2.934 and the new price is S_{UU}=\$18.81. The bottom branch represents the possibility of a low return, -12.4%. The new log price is 2.494 and the new price is S_{UD}=\$12.11. From the bottom node at t=1, the tree also branches twice. The top branch once again represents the possibility of a high return, U=31.6%. The new log price is 2.494 and the new price is S_{DU}=\$12.11. The bottom branch represents the possibility of a low return, -12.4%. The new log price is 2.054 and the new price is S_{DD}=\$7.80.

Because of the way we selected the returns at each branch, the stock price at the two interior nodes at t=2 are identical: $S_{UD} = S_{DU}$. Therefore, we can represent the same dynamics in the form of a recombining tree as in Figure 6.10. A recombining tree is computationally convenient since the number of nodes grows linearly with the number of period, whereas in the non-recombining tree the number of nodes grows exponentially. However, there will be cases in which a recombining tree is not useful since information about the history of returns is lost. More on this later.

At first glance, a binomial tree appears too crude to adequately represent the risk structure we used earlier to model a stock price. But the tree can be expanded to include many more steps, and in that case the final branchings are a very dense representation of the range of possible future stock prices. Figure 6.11 shows a recombining binomial tree with N=24 time steps.

In the illustrations shown so far, we carefully selected the high and low returns for the purpose of generating stock returns that are approximately normally distributed. Figure 6.11 shows the probability distribution across the final node of the tree. You can see that this distribution is approximately normal. There are several different ways to define the returns and probabilities along the tree so as to assure that the distribution across nodes is approximately normal. In these lecture notes our default method will be to set the up node as one standard deviation above the expected value and the down node as one standard deviation below the expected value.[1] The probability of following either path will be ½. In this example, with μ=9%, σ=22%, and Δt=1 we have:

$$E[\ln(S_1)] = \ln(S_0)+(\mu-{}^1/_2\sigma^2)\Delta t = 2.398,$$

so that

$$\ln(S_U)= \ln\left(S_0\right)+\left(\mu-\tfrac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t} = 2.618$$

$$\ln(S_D)= \ln\left(S_0\right)+\left(\mu-\tfrac{1}{2}\sigma^2\right)\Delta t-\sigma\sqrt{\Delta t} = 2.178.$$

We repeat this process at the subsequent nodes. Starting from t=1, at the top node where S_1=S_U, we have:

$$E[\ln(S_2)] = \ln(S_U)+(\mu-{}^1/_2\sigma^2)\Delta t = 2.714,$$

so that

[1] This is slightly different from the more well known Cox-Ross-Rubinstein method. For finite length of the period, the choice of method matters. However, as the length of a period decreases and the number of steps per year increases, both methods converge to the normal distribution and calculations done using the two methods yield the same result.

$$\ln(S_{UU}) = \ln(S_U) + \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} = 2.934$$

$$\ln(S_{UD}) = \ln(S_U) + \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t} = 2.494.$$

Applying these steps throughout the tree produces a probability distribution across returns at each date that is approximately normal. As the length of the period, Δt, declines to zero, the distribution approaches the normal distribution. The mean of this distribution of returns grows linearly with the time horizon. The variance of this distribution grows linearly with the time horizon so that the standard deviation grows as the square root of the time horizon. These are the same properties that we derived for our discrete stochastic process and for the continuous time stochastic process. The binomial tree is just another tool for representing the same dynamics.

6.9 Complications

The assumption of normality in returns and lognormality in stock prices leads to a very tractable and convenient model. This is a valuable property. But does the model fit the data? Are stock returns normally distributed? Are stock prices lognormal?

The lognormal model has proven very valuable for stock prices. At a certain level of precision, it appears to work very well. However, there appear to be a number of ways in which stock prices don’t fit the model. One of the most important ones is the observation of fat tails, i.e. more large returns – whether positive or negative – than predicted by the normal distribution. How large of a problem this is depends upon the purposes to which one is employing the model. For large investment funds this is an important detail to consider. We will come back to this issue later in the lecture notes.

Another, more implicit assumption we have been making as we developed the model above is that the parameters were stable throughout the horizon we were studying. This need not be the case. In particular, it is widely observed that there are periods of lower and periods of higher volatility. Although it is convenient to develop the model with a constant mean and volatility, it is not necessary that they be constant. Allowing the mean and the volatility to change through time will certainly complicate our calculations. For example, in the binomial tree, the ability to have the tree recombine is predicated on this stability in the parameters. If the parameters are changing, we can develop a ‘fix’ to our modeling, but it will require more attention and time.

6.10 From Factor Risk to Project Risk

Having modeled the price of a stock and measured its risk, we now want to analyze the risk of a derivative claim on the stock. The risk of the stock flows through to the derivative claim. But the risk is altered by the nature of the derivative claim. Sometimes the derivative may be riskier, and sometimes the derivative may have less risk. The source of the risk is the stock price risk. In that sense, we call the stock price the factor. But the factor risk can be either concentrated or diluted by the terms of the derivative claim. And the amount of factor risk channeled through to the derivative claim may change with the price of the stock.

While we focus, for the moment, on a derivative claim on a stock, this illustrates a very general principle in risk analysis. Project cash flows reflect underlying factors. The key question in analyzing project risk is understanding (i) the risk of the factor, and (ii) how that risk is channeled to the project. Going from the underlying factor risk to project risk can be complicated, but it is essential. Analyzing a simple derivative claim on a stock price is a convenient starting point for developing our understanding of this complicated phenomenon.

We focus on the risk of a call option. A call option written on a stock gives the option holder the right to buy the stock at a fixed price, the exercise price any time up to an including a maturity date. Later we will examine carefully how the price of a call option is determined. For the moment, we will simply take as given that the price of the call is determined by the Black-Scholes equation:

$$C = S\,N(d_1) - K\,e^{-rT}\,N(d_2),$$

where,

$$d_1 = \frac{\ln(S) - \ln(K) + \left(r + \tfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S) - \ln(K) + \left(r - \tfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

and where,

C is the price of the call, S is the current price of the stock, K is the exercise price, T is the maturity date, σ is the volatility of the stock, r is the risk free rate of interest, and $N(\,)$ is the cumulative normal distribution.

The top panel of Figure 6.12 shows the graph of the call price as a function of the stock price. For this calculation we have set the exercise price to $50, the maturity to 1 year, the volatility to 30% and the risk-free interest rate to 5%. The key fact to notice is that the graph is not linear. Only as the stock goes deep in-the-money, i.e. the stock price is above the exercise price, does the function become approximately linear. When the stock price is below the exercise price so that the call option is out-of-the-money, the function is very convex. One consequence of this relationship is that the riskiness of the call option changes as the stock price changes. We will write σ_{option} to denote the volatility of the derived option price, and σ_{stock} to denote the volatility of the underlying stock. At very high stock prices, when the option is deep in-the-money, holding the call is virtually identical to holding the stock, so that the risk of the call option equals the risk of the stock: $\sigma_{\text{option}} \cong \sigma_{\text{stock}}$. At very low stock price, when the option is far out-of-the-money, the risk of the option grows far above the risk of the stock: $\sigma_{\text{option}} > \sigma_{\text{stock}}$.

The formula relating the risk of the option to the risk of the stock is:

$$\sigma_{\text{option}} = \sigma_{\text{stock}} \times |\Omega| = \sigma_{\text{stock}} \frac{S\,\Delta}{C} = \sigma_{\text{stock}} \frac{S\,N(d_1)}{C}$$

where Ω is the option elasticity and Δ is the option delta, the measure of the change in the option price for a $1 change in the stock price. The bottom panel of Figure 6.13 shows the graph of the volatility of the call price as a function of the stock price. Recall that the volatility of the stock price is constant, regardless of the level of the stock price. At very high stock prices, the volatility of the call is almost double the volatility of the stock. As the stock price drops, the volatility of the call price grows, and grows sharply, so that the volatility of a far out-of-the-money option grows to nearly ten times the volatility of the stock.

This is a good illustration of how important it is to understand the relationship between the risk of the underlying factor and the risk of the derivative asset. One underlying factor for a commodity producer – a gold mining company, for example – is the price at which it can sell the commodity. If the company will produce the gold regardless of how low the price goes, then its payoff will be linear with the gold price. Suppose instead that the company has flexibility to respond to changes in the gold price. Suppose it will shut down some production as the price of gold falls, and if the company owns some mines that it will reopen or potential mines that it will develop should the gold price rise sharply, then the company's payoff will be non-linear in the price of gold, and the company's risk will change as the gold price changes just as the call option risk changes as the stock price changes.

6.11 Application of the Random Walk to Other Variables

The use of stochastic processes in the field of finance was pioneered through the development of the geometric Brownian motion model applied to stocks. As the power of this tool became clear, it also occurred to many analysts that this tool could be applied to modeling many other variables. Unfortunately, the widespread application of this model to other variables reflected the adage, "when the only tool you have is a hammer, everything looks like a nail." In the following section, we will discuss alternative stochastic processes and their application to modeling appropriately selected variables. Nevertheless, there are some variables where this simple model has, perhaps, been usefully applied.

One such case is the price of gold. This is because gold is used as an investment vehicle, and because the cost of storing gold is relatively small. For example, in analyzing the hedging strategies of gold companies, Fehle and Tysplakov (2005) model the gold price based on data from 1992-2000 setting the mean annual return at 2% and the annual volatility at 10%.[2]

d'Halluin et al. (2003) examine how to determine when to expand bandwidth capacity on a wireless network when the growth in traffic is modeled as a geometric Brownian motion.[3]

The random walk model is a very specialized process with particular properties that make it attractive for modeling stock prices. One such feature is the property that the uncertainty about future stock price grows without bound. There is no maximum beyond which the price is certain not to go.

A second key feature of this model is that changes in the current spot price translate one-for-one into a permanent revision of the forecast of future prices. There is no such thing as a temporary shock to the stock price, i.e., a shock in which the stock price goes up and then predictably comes down again.

[2] Fehle, F. and S. Tysplakov, (2005), Dynamic risk management: Theory and evidence, *Journal of Financial Economics* 78, 3–47.

[3] d'Halluin, Y., P.A. Forsyth, and K.R. Vetzal, 2003, "Wireless Network Capacity Investment," Working Paper.

The two panels of Figure 6.13 illustrate this second key feature, the permanent impact of any shock in a random walk process. The top panel shows a historic sample of data for a price series. The price has recently experienced a sharp run-up. If we were to forecast the future path of the price using the random walk model, our forecast would follow the nearly straight line heading to the right and slightly upward. This line reflects the expected rate of growth in the price, μ. The forecast takes off from wherever is the most recent value. Any further run-up in the price above the expected growth rate will shift the whole forecast line up, at all horizons. This is the sense in which any shock or innovation to the price is treated as a permanent shock.

The bottom panel of Figure 6.13 shows the same historic price data, but constructs a forecast of future prices by applying a different model, the mean reverting model. The mean is shown as a dashed line running through the historic data and continuing out into the future time. The most recent run-up in the historic price series has sent the price far above the mean. In the mean reverting model, we expect the price to return back to the mean. The forecast is shown as the curved line heading down from the most recent historic price towards the mean and ultimately asymptoting to the mean. The mean price is growing at a rate μ, but the actual price is forecasted to be declining until it returns close enough back to the mean. Therefore, the recent run-up in price is viewed exclusively as a temporary shock. The long-run forecast is entirely unaffected by the shock. Only the short-run forecast is dragged upward.

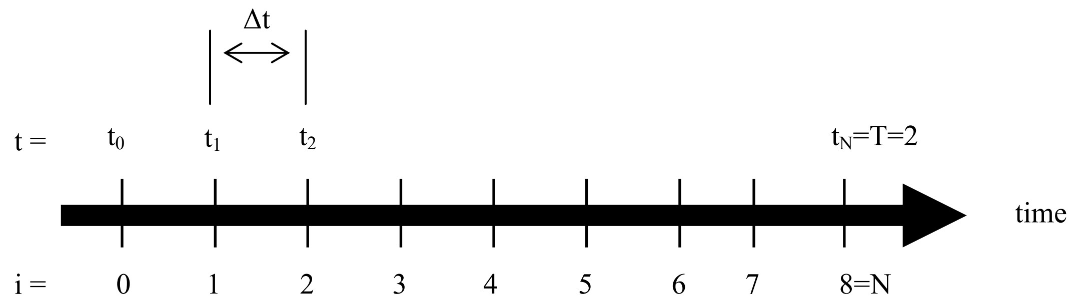
This property of the random walk makes the model very easy to handle. Not only are the forecasted distributions normal variables, but because of this “permanent shock” property, the conditional distribution always have the same structure, no matter the current value of the variable. This makes working with the data and estimating the parameters very simple. There are never any complicated adjustments to be made. In a mean-reverting model, the conditional distributions are changing depending upon the current value of the variable. This makes working with the data and estimating parameters more difficult since adjustments have to be made to compensate or reflect these changing conditional distributions.

We can illustrate the difference with the simple example of reporting volatility on an annualized basis. Suppose we measure returns using weekly data and calculate a weekly volatility. The annualized volatility is calculated by taking the weekly volatility and multiplying by the square root of 52. This makes sense with the random walk because the volatility grows by the square root of time, and the volatility in the process never changes. The annualized weekly volatility should roughly match what you would get if you calculated the volatility from annual data. But the same is not true with a mean reverting process. Annualizing the weekly volatility by multiplying it by the square root of 52 will produce a number that is larger than what you would get if you calculated the annual volatility from annual data. With a mean reverting process, one needs to take more care in this type of reporting and comparison.

Unfortunately, the 'permanent shock' property is exactly what makes the random walk model unattractive for representing the dynamics of many other variables that determine project cash flows and values. These factors include interest rates, foreign exchange rates, various commodity prices such as oil, natural gas and electricity, and many others. Many of these other factors exhibit more complicated conditional forecasts. Many exhibit mean reversion of one form or another, for example. In most cases it is a mistake to casually move from returns and volatility measured and denominated over one interval of time to returns and volatility measured and denominated over another interval. So it is necessary to develop other models appropriate to the peculiar dynamics of these other, underlying factors central to asset valuation and management.

Having already alluded to the properties of a mean reverting process, it is now time to formally present one. The next part of this chapter develops the pure mean reverting process and shows how it is used to model interest rates. The final part of the chapter gives a brief overview of other processes that are used to model a wide range of commodity prices and other factors.

Figure 6.1
Period Layout for a Monte Carlo Simulation



Time horizon, $T=2$ years,
Number of periods in total, $N=8$,
Number of periods per year, $n=N/T=4$,
Length of a period, $\Delta t=1/n=0.25$, i.e., quarterly.

Time horizon, $T=2$ years,

Figure 6.2
Simulations of the Same Horizon Using Different Period Lengths

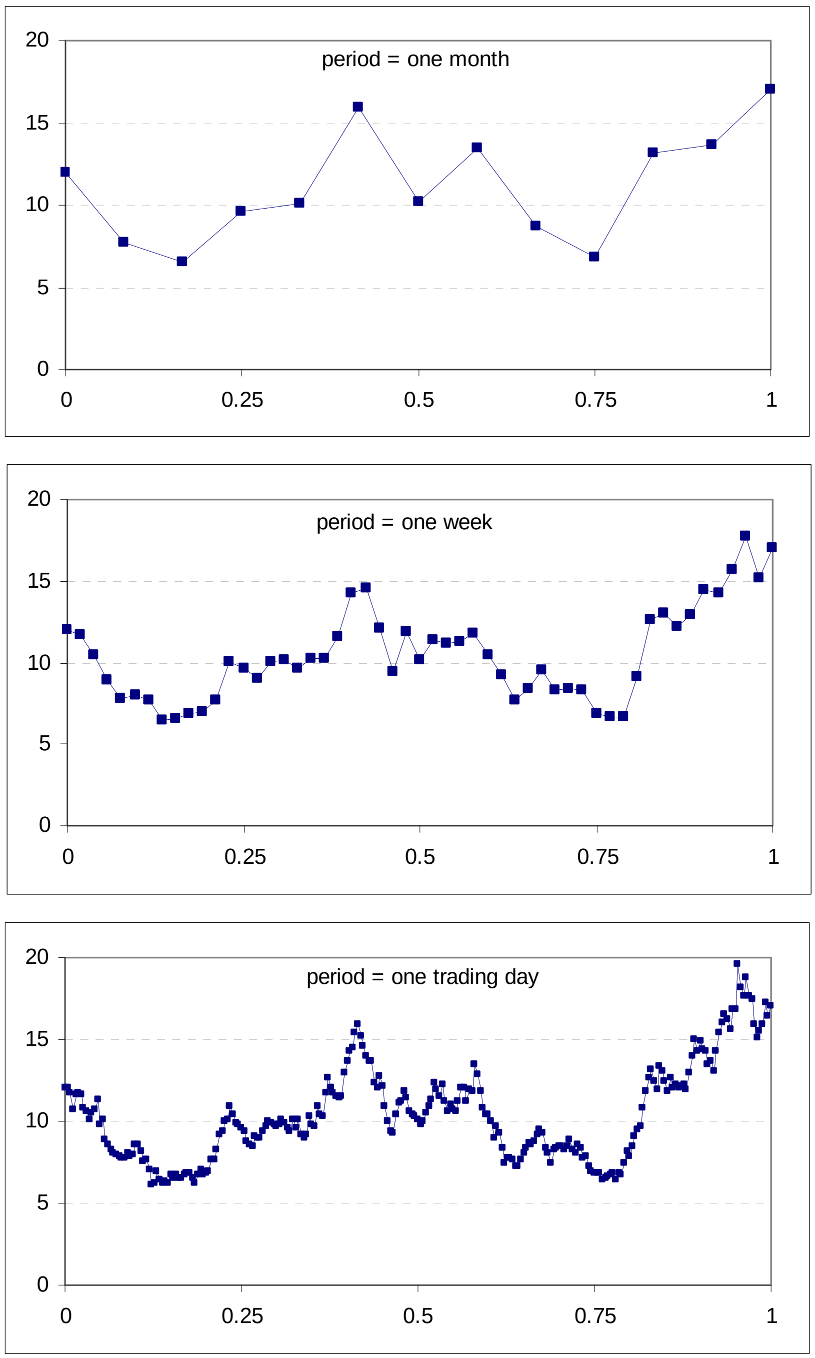


Figure 6.3
Four Sample Paths of the Same Simulation

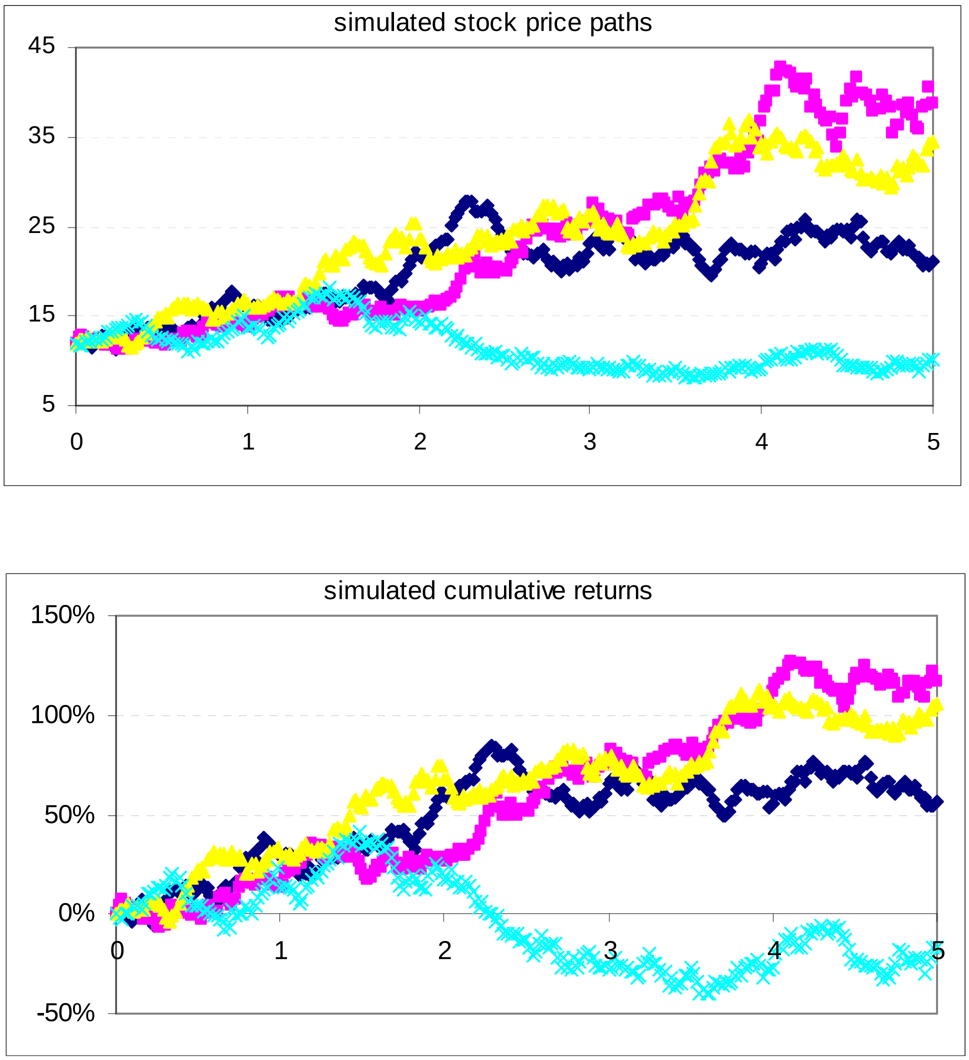


Figure 6.4
Histograms for a Sample of 100 Paths

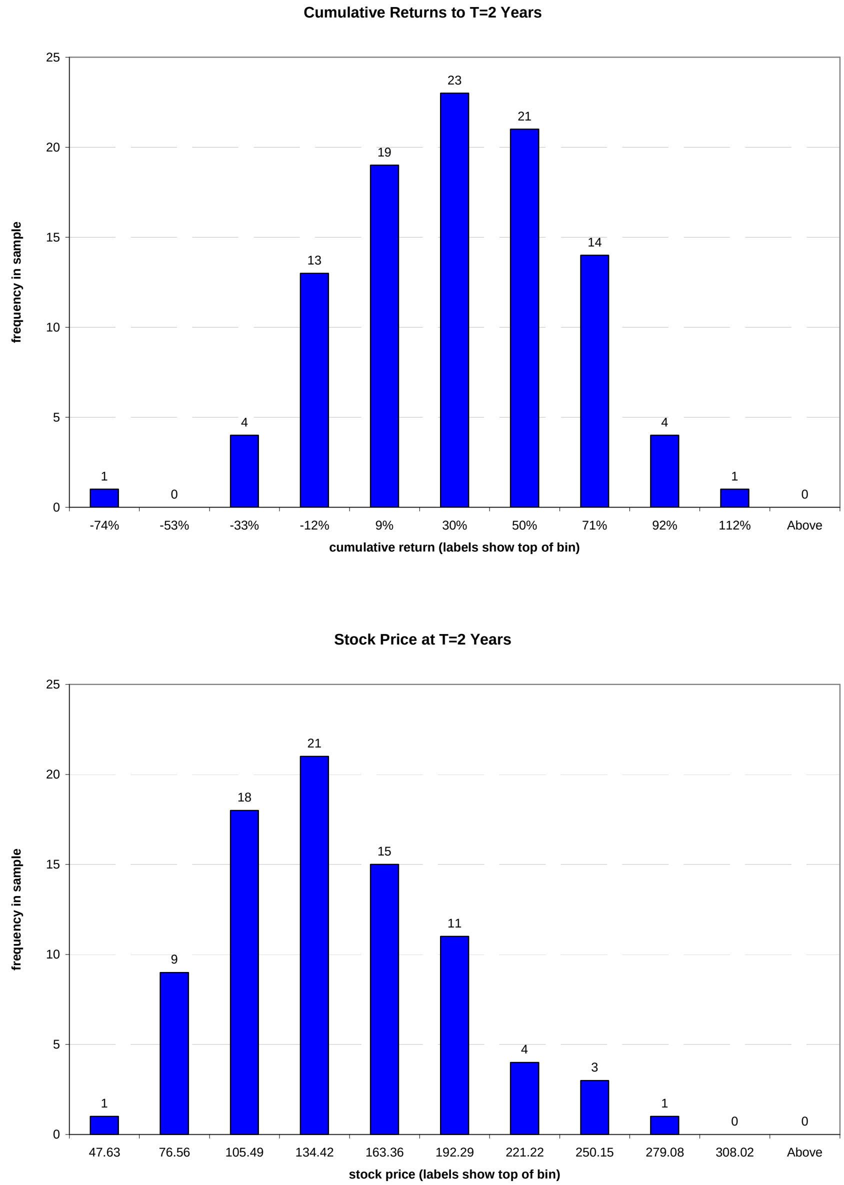


Figure 6.6
Expected Values with Confidence Bounds

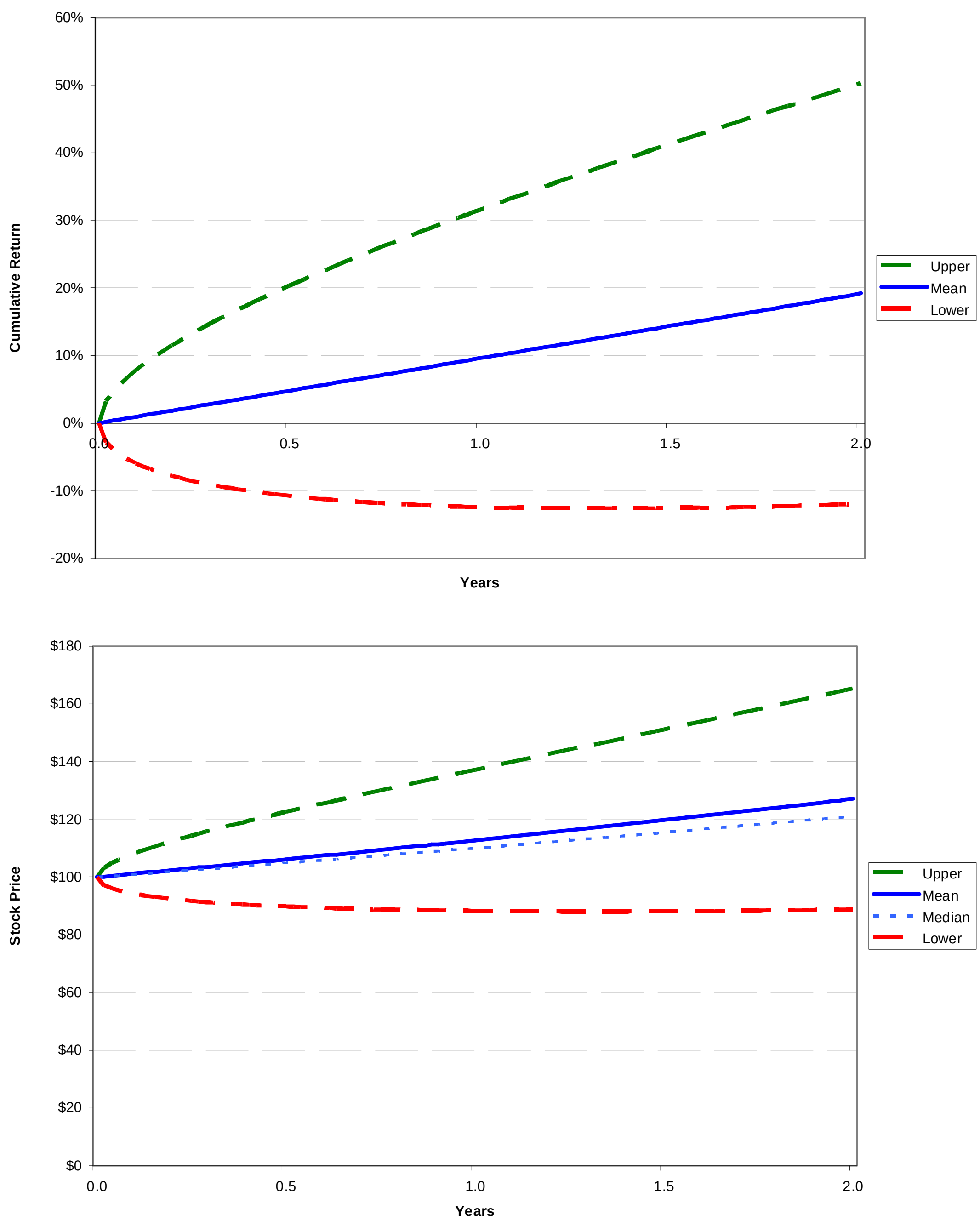


Figure 6.7
One-Step in a Binomial Tree

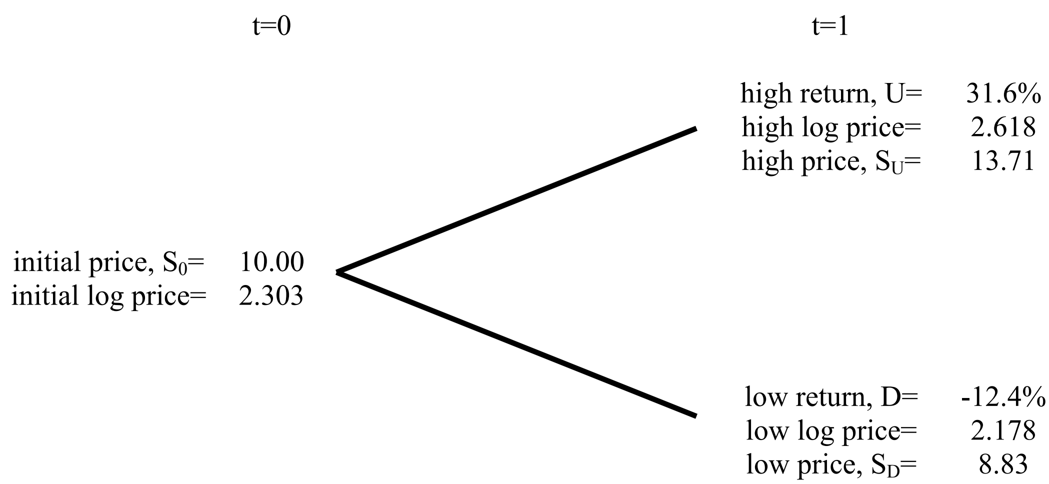


Figure 6.9
Two Steps in a Binomial Tree

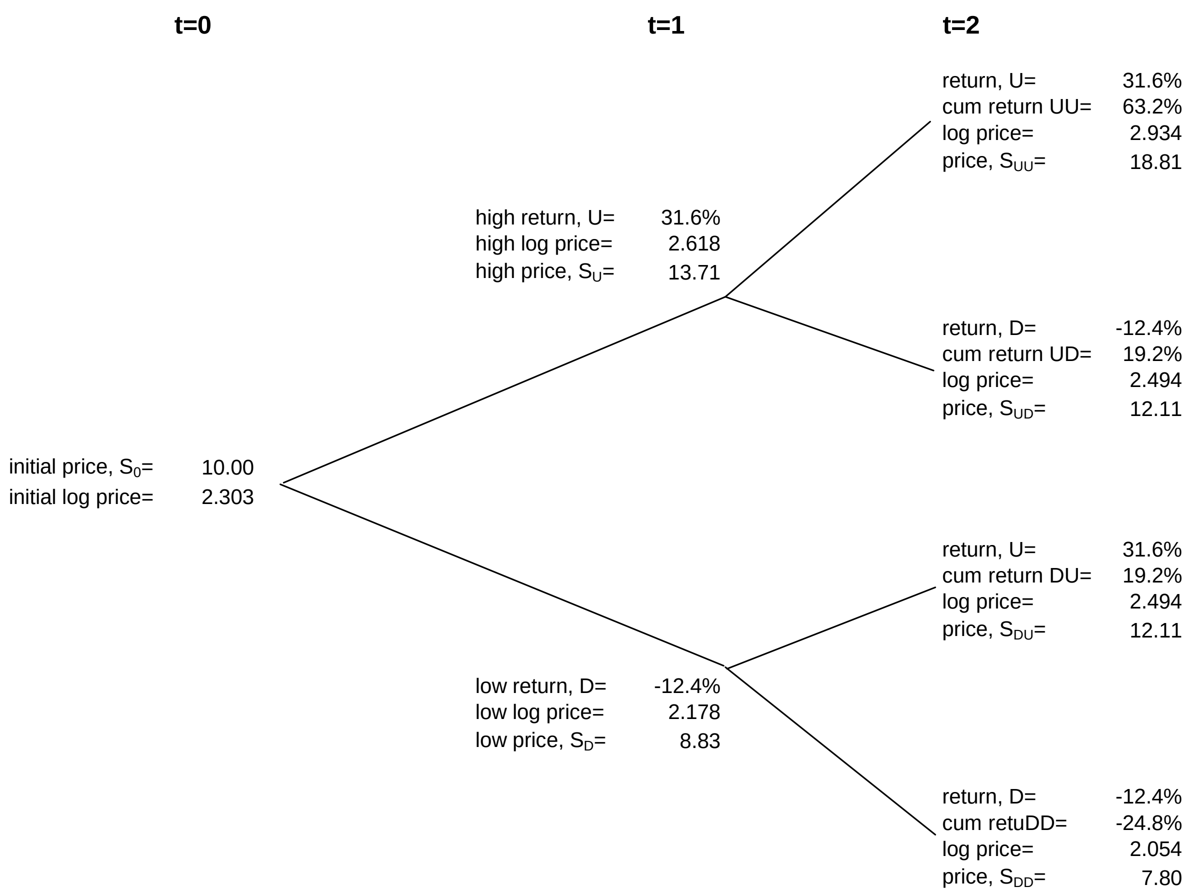


Figure 6.10
Recombining in a Binomial Tree

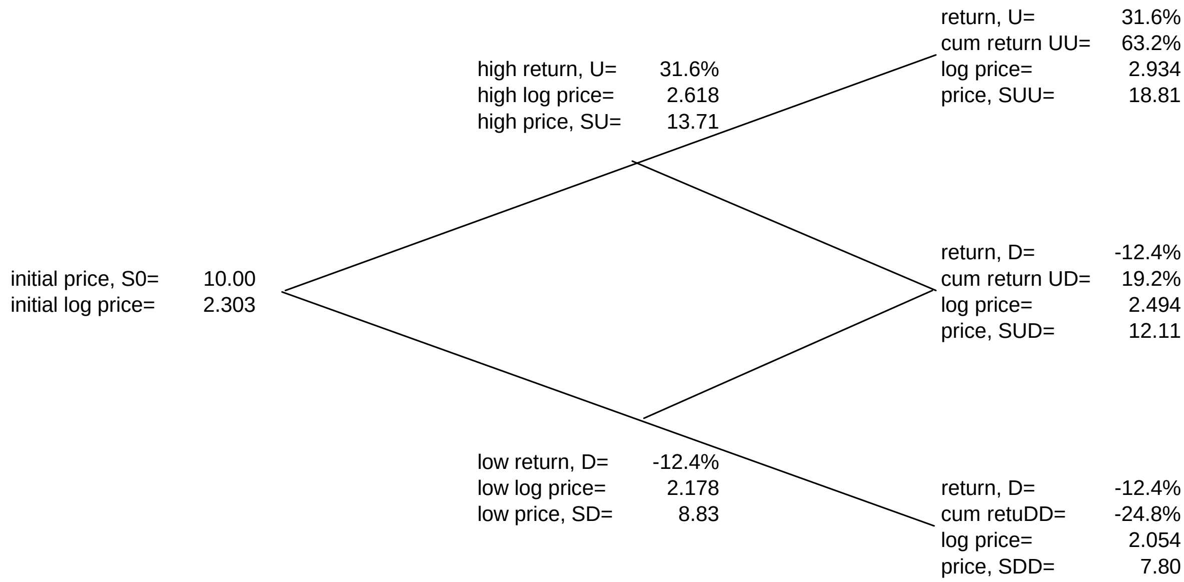


Figure 6.11
A 24 Period Recombining Binomial Tree

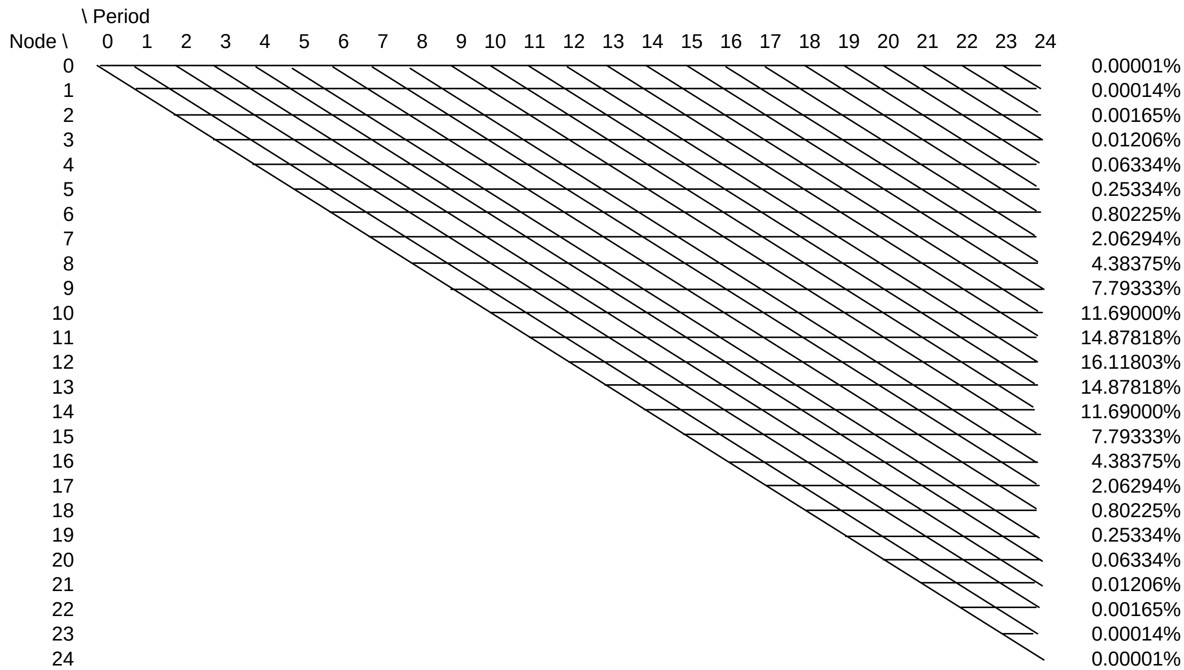


Figure 6.12
Call Price and Call Volatility as a Function of the Stock Price

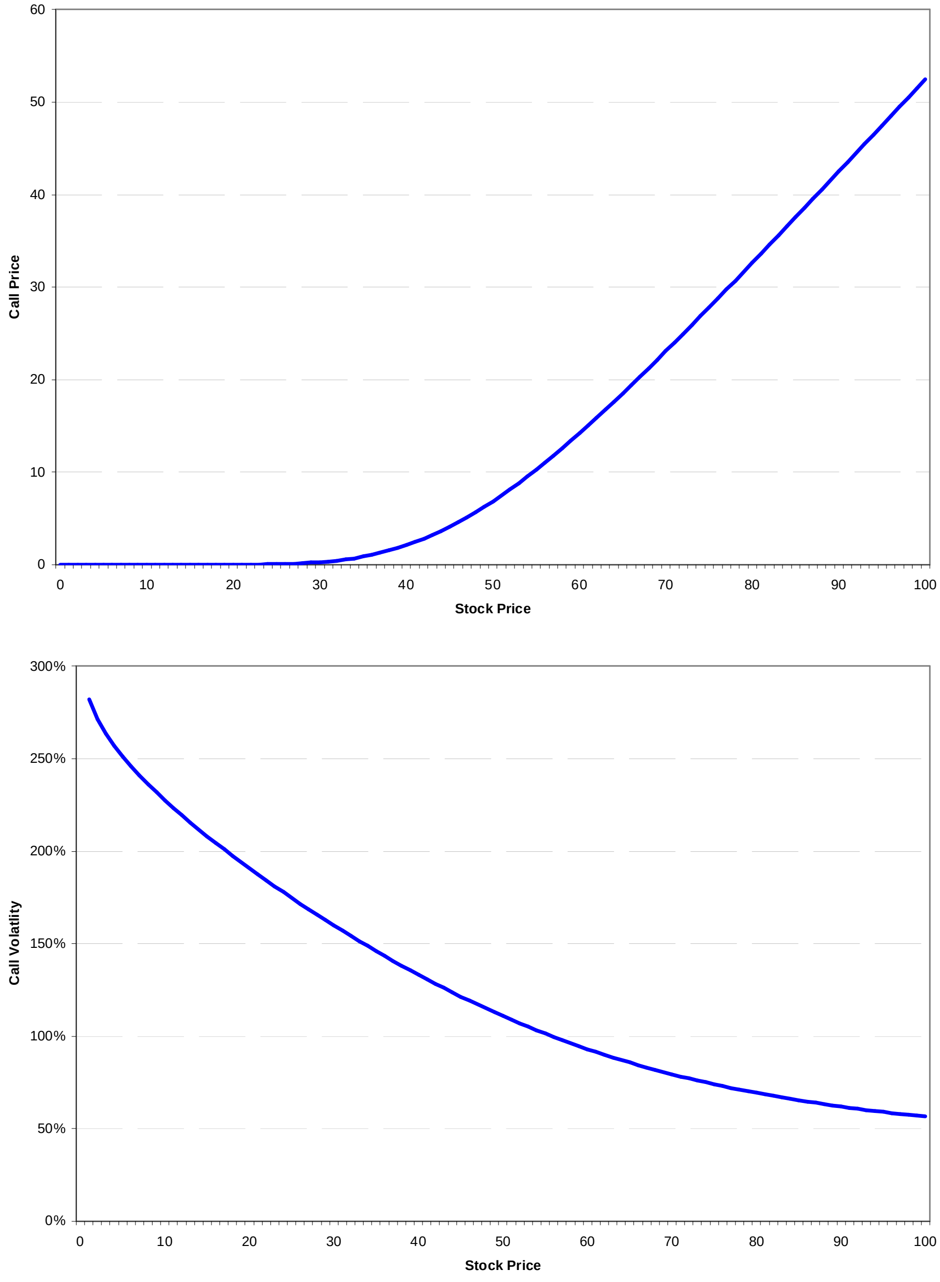


Figure 6.13
Forecast Made Using a Random Walk Model
vs.
Forecast Made Using a Mean Reverting Model

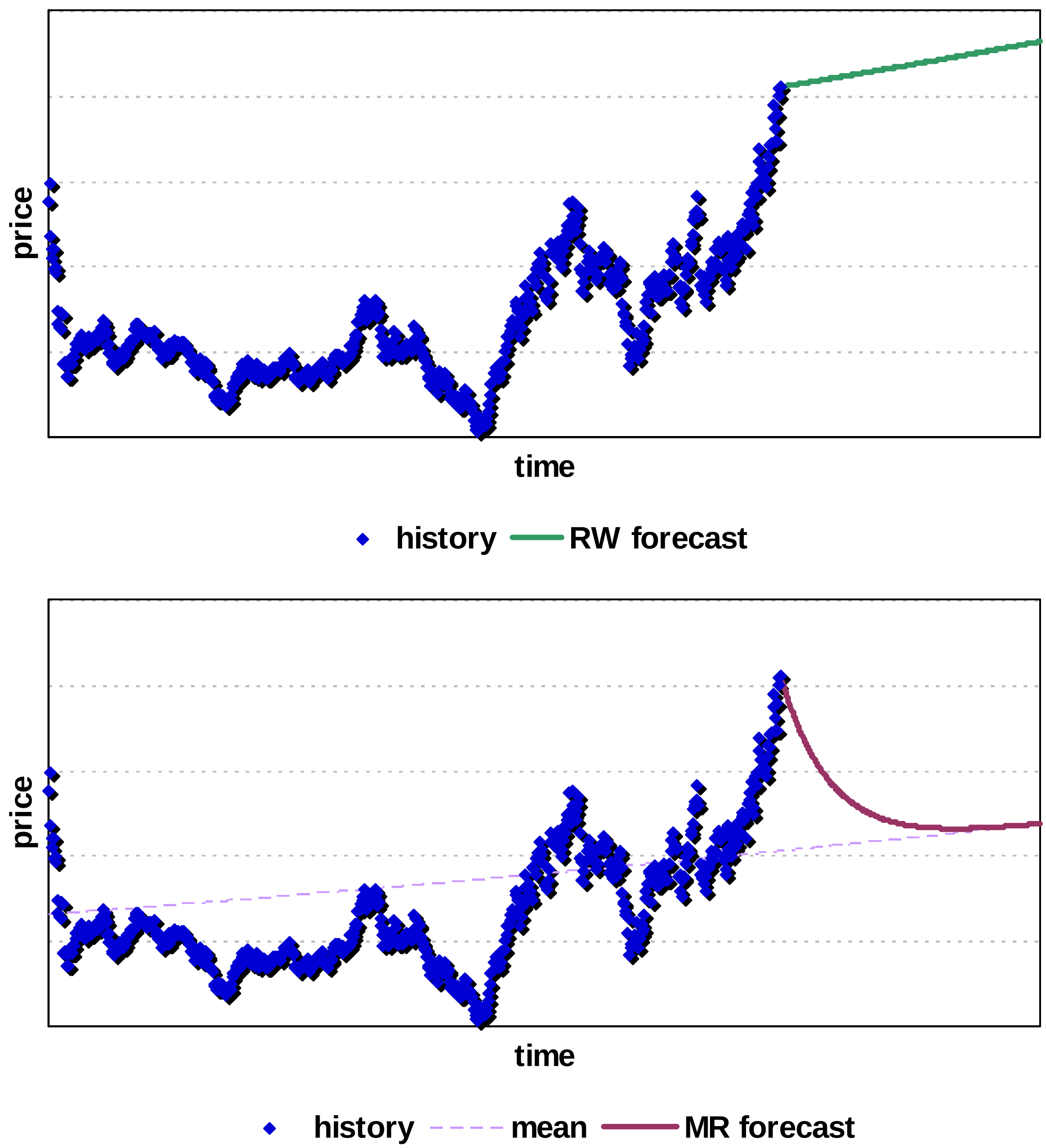


Table 6.1
Calculation of a Sample Path

week	random variable	return	cumulative return	stock price
0				12.00
1	1.57580	4.99%	4.99%	12.61
2	0.34769	1.24%	6.24%	12.77
3	-2.42436	-7.21%	-0.98%	11.88
4	-0.01376	0.14%	-0.83%	11.90
5	-0.94792	-2.71%	-3.54%	11.58
6	1.72028	5.43%	1.89%	12.23
7	0.14646	0.63%	2.52%	12.31
8	1.39280	4.43%	6.96%	12.86
9	-0.34718	-0.87%	6.08%	12.75
10	-0.98923	-2.83%	3.25%	12.40

Table 6.2
Estimating Drift and Volatility Parameters

week	stock price	return
0	12.00	
1	12.61	4.99%
2	12.77	1.24%
3	11.88	-7.21%
4	11.90	0.14%
5	11.58	-2.71%
6	12.23	5.43%
7	12.31	0.63%
8	12.86	4.43%
9	12.75	-0.87%
10	12.40	-2.83%
	Mean	0.32%
	Std. Dev.	3.99%

Variable	Estimate	Input	Error
μ	21.0%	12.0%	75.2%
σ	28.8%	22.0%	30.7%