Combinatorics: The Fine Art of Counting

Week Three – Seasonal Menu

This week's menu is divided into two sections. The first part is a set of appetizer problems which can all be solved using a straight-forward application of concepts covered in class. The second part consists of four seasonal menus (winter, spring, summer, and fall) each of which contain problems we will be working on during next week's class. After sampling a few appetizers, you should try some of the problems from the menu corresponding to the season of your birthday. Think about and/or solve as many of these problems as you can. Any time spent thinking about these problems is worthwhile, even if you don't solve them. It will help prepare you for next week's class which will be pretty dull if you haven't looked at the problems. If you have an appetite for more, feel free to browse the other season's menus as well.

Appetizers

1. How many 5 letter words have at least one double letter, i.e. two consecutive letters that are the same?

2. A Venn diagram is drawn using three circular regions of radius 1 with their centers all distance 1 from each other. What is the area of the intersection of all three regions? What is the area of their union?

3. A diagonal of polygon is any line segment between vertices which is not an edge of the polygon. How many diagonals does an n-sided polygon have?

4. A diagonal of a polyhedron is any line segment between the vertices of a polyhedron which is not an edge of the polyhedron. A tetrahedron has no diagonals, while an octahedron has 3 diagonals. How many diagonals do the cube, dodecahedron and icosahedron have?

5. Let P_n denote the n^{th} *centered pentagonal number*: the number of points in a pentagonally symmetric lattice with one point in the center and n points on each side. The first four pentagonal numbers are 1, 6, 16, and 31. Find a formula for P_n.

6. Let A_1, A_2, A_3 be sets contained in the universe U of all bit strings of length 3, where A_i is the set of all strings in U with a 1 in the i^{th} position. For example A_2 = { 010, 011, 110, 111 }. Given that $A_1 \cup A_2 \cup A_3 = \{111\}$, and $A_1 U A_2 U A_3 = (\{000\})^c$, use the principle of inclusion/exclusion to compute the size of the sets $A_i U A_j$ (note the size does not depend on i and j).

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Winter

1. How many different even integers ≥ 4000 and < 7000 have four different digits? (AIME 1993 #1)

2. How many integers less than 500 can be written as the sum of 2 positive cubes?

3. Two of the squares of a 7x7 checkerboard are painted yellow and the rest are painted green. Two color schemes are indistinguishable if the board can be rotated so that they look the same. How many distinct color schemes are there? (AIME 1996 #7)

4. Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has a face on one side and tails on the other. How many distinguishable stacks of the eight coins have no coins stacked face-to-face? (AIME 2005 #5)

5. Prove that it is impossible to draw a general Venn diagram for 4 sets with circles, i.e. 4 circles representing sets cannot be drawn in the plane so that every possible intersection of the 4 sets has its own distinct region with non-zero area.

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Spring

1. How many diagonals of a decagon (a 10 sided polygon) are not parallel to any of the sides (AMC12)?

2. Let *p(m,n)* denote the n^{th} centered m-agonal number: the number of points in an m-sided polygonal lattice with a point in the center and n points per side. Take p(m,1) to be 1 (i.e. start with a single point) and then add concentric polygons with k equally spaced points for $2 \le k \le n$. Note that *p(3,n)* is not the same as the n^{th} triangular number, nor is *p(4,n)* the same as the nth square number - these numbers are based on lattices constructed starting from a corner rather than the center. Find a general formula for *p(m,n)* that works for any m and any n.

3. In a shooting match a marksman must break eight targets arrange in three hanging columns of 3, 3, and 2 targets respectively. Whenever a target is broken, it must be the lowest unbroken target in its column. In how many different orders can the eight targets be broken? (AIME 1990 #8)

4. An integer is called *snakelike* if its decimal representation has consecutive digits alternately increasing and decreasing (e.g. 192837465 is snakelike). How many 4 digit numbers with distinct digits are snakelike? (AIME 2004 #6)

5. A deck of forty cards consists of four 1's, four 2's, …, and four 10's. One matching pair of cards is removed from the deck. Two cards are now drawn at random from the deck. What is the probability they form a pair? (AIME 2000B #3)

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Summer

1. A convex polyhedron P has 26 vertices, 60 edges, and 36 faces, 24 of which are triangular and 12 of which are quadrilaterals. A space diagonal is a line segment connecting two non-adjacent vertices that do not belong to the same face. How many space diagonals does P have? (AIME 2003 #3)

2. The nine horizontal and nine vertical lines on an 8x8 checkerboard form *r* rectangles and *s* squares. Find *r* and *s*. (AIME 1996 #1)

3. The increasing sequence 2, 3, 5, 6, 7, 10, 11,… consists of all positive integers which are neither the square nor the cube of a positive integer. Find the 500th term of this sequence. (AIME 1990 #1)

4. Show why three-of-a-kind beats two-pair in poker by counting the number of poker hands containing exactly three cards of the same rank (with the other two cards different ranks) versus the number of poker hands containing two different pairs of cards with the same rank (but no three of the same rank).

5. Ten points in the plane are given, no three co-linear. Four distinct segments joining pairs of these points are chosen at random with uniform probability. What is the probability that three of the four segments chosen form the sides of a triangle? (AIME 1999 #10)

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Fall

1. Ten male/female couples meet for a dinner party and they all greet each other in the following manner: the men all shake hands, the women all exchange kisses, and each man exchanges kisses with each woman. Assuming every person greets every other person and counting each handshake or kiss exchange between two people just once, how many handshakes and kiss exchanges are there? How many greetings all together? Given that the number of greetings is the sum of the handshakes and the kisses, can you derive a general combinatorial identity?

2. Ten points are marked on a circle. How many distinct convex polygons can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.) (AIME 1989 #2)

3. A fair coin is tossed ten times. Find the probability that heads never occurs on two consecutive tosses (AIME 1990 #9)

4. One hundred concentric circles with radii 1,2,3,…,100 are drawn in the plane. The inner circle is colored red and each region bounded by concentric circles is colored green or red with no two adjacent regions the same color. What fraction of the entire circle with radius 100 is colored green? (AIME 2003 #2)

5. Given n lines in the plane in general position (each line intersects every other line in a distinct point), into how many regions to they divide the plane?

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Summer 2007