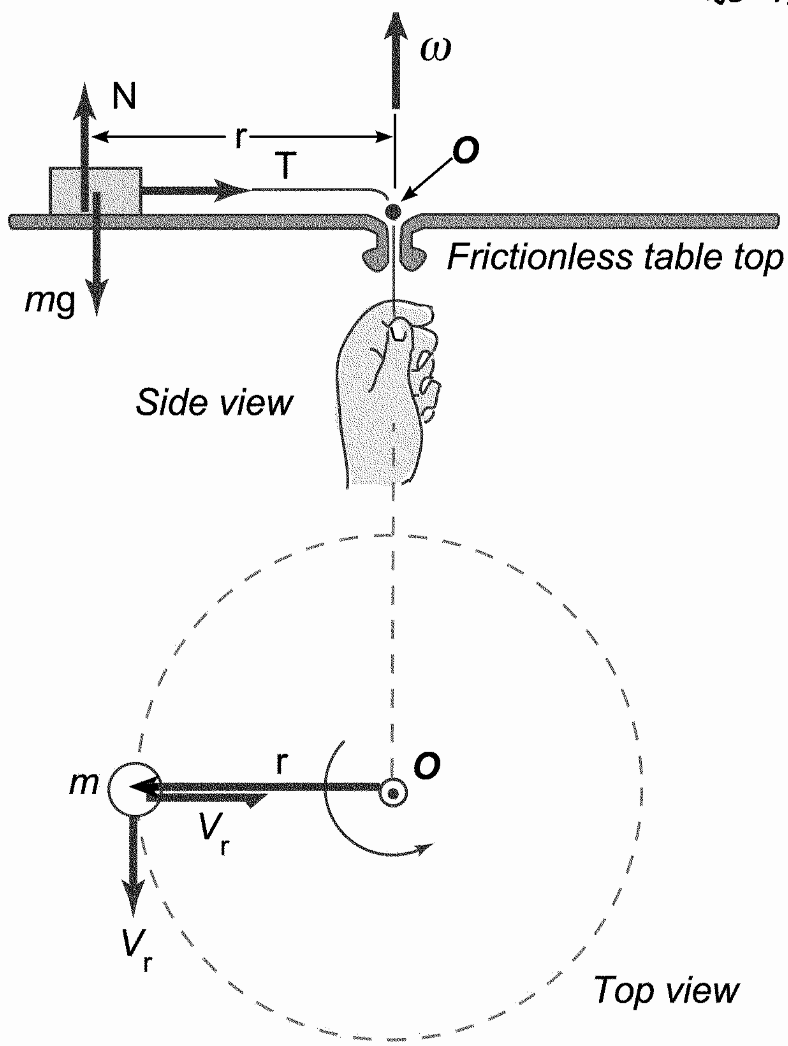
Example

Particle on a string at radius r_1 is moving in a circle with angular velocity ω_1.

String shortened to new radius r_2.
What is new angular velocity ω_2?



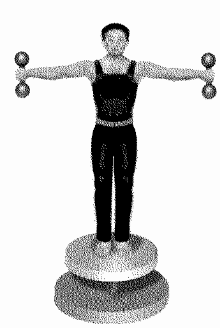
Ans: shortening string requires a radial force only on particle — no torques.

$\therefore \quad L_i = L_f \quad$ [Ang. Mom. Conserved]

$$I_i \omega_i = I_f \omega_f$$

$$\cancel{m} r_1^2 \omega_i = \cancel{m} r_2^2 \omega_2$$

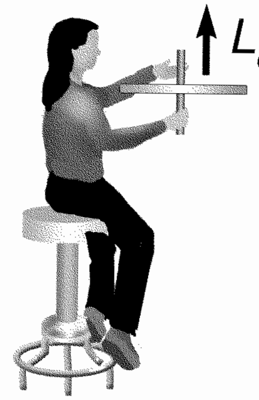
$$\omega_2 = \left(\frac{r_1}{r_2}\right)^2 \omega_1$$



Conservation of angular momentum about a fixed axis.

Ex: skaters, gymnasts, ballet dancers change ang. velocity by altering moment-of-inertia by extending arms etc.

Example: Spinning Bicycle Wheel



Student holds a spinning bicycle wheel. Initially student and stool at rest. Wheel spins in horizontal plane with initial ang. mom. $\vec{L_o}$ up.

Wheel is inverted through center by 180°. What happens?

System = student + wheel + stool.

Total Ang Mom = $\vec{L_o}$ [wheel only spinning]

As wheel is inverted, student supplies a torque — it is internal to the system.

No external torques acting on the system about vertical axis. Free spinning axel.

$\therefore$ Total angular momentum is conserved.

Initial: $\vec{L}_{system} = \vec{L_o}$

Finally: $\vec{L}_{system} = \vec{L}_{student+stool} + \vec{L}_{wheel}$

$$\therefore \vec{L_o} = \vec{L}_{student+stool} + \vec{L}_{wheel}$$

$$\vec{L_o} = \vec{L}_{st+stool} - \vec{L_o}$$

$$\therefore \vec{L}_{st+stool} = 2\vec{L_o}$$

If I_p = moment-of-inertia (student + stool)

$$\therefore \quad I_p \omega_p = 2 I_o \omega_o$$

If wheel is held at an angle θ measured from the vertical axis, the student acquires an ang. mom given by

$$L_o (1 - \cos\theta).$$

In this case the ang. mom. precesses about axis of rotation of platform. A horizontal torque supplied by the student then accounts for this precession.

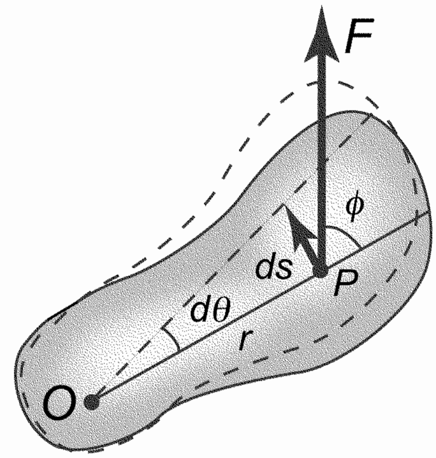
Work and Energy in Rotational Motion

A force applied to a moving body does work on the body.

The work done by the force $\vec{F}$ as the body rotates through a small distance

$ds = r\,d\theta$ in a time dt is

$$dW = \vec{F}\cdot\vec{ds} = (F\sin\phi)\,r\,d\theta$$



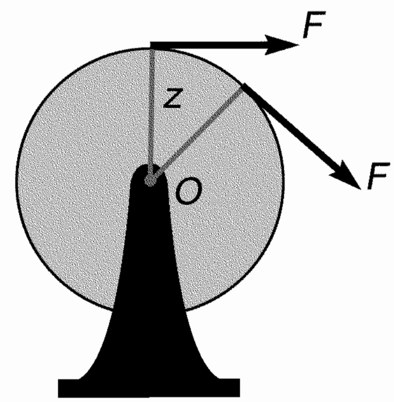
A rigid body rotates about an axis through *O* under the action of an external force applied at *P*.

$F\sin\phi$ = tangential component of $\vec{F}$.

Radial component does no work since it is $\perp$ to $\vec{ds}$.

$$(F\sin\phi)\,r = \tau_z$$

(Torque about axis of rotation)



A force applied to a rotating body does work on the body.

$$dW = \tau_z\,d\theta$$

$$W = \int_{\theta_1}^{\theta_2} \tau_z\,d\theta \qquad = \tau_z\,\Delta\theta \text{ for constant torque}$$

Power

$$P = \frac{dW}{dt} = \tau_z\frac{d\theta}{dt} = \tau_z\,\omega$$

Work-Energy Theorem in Rotational Motion

The work done by the torque changes the KE of the body.

$$\tau \rightarrow \alpha \qquad \omega_1 \rightarrow \omega_2$$

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

$$\tau\, d\theta = dW = I\omega\, d\omega$$

Total work done is

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta' = \int_{\omega_1}^{\omega_2} I\omega' d\omega' = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

Angular velocity: $\omega_1 \rightarrow \omega_2$
Angular displacement: $\theta_1 \rightarrow \theta_2$

$$W = \Delta K = K_f - K_i = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2.$$

If the force acting is conservative (gravity/spring) then the work done is the negative of the change in PE.

$$-U_2 + U_1 = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

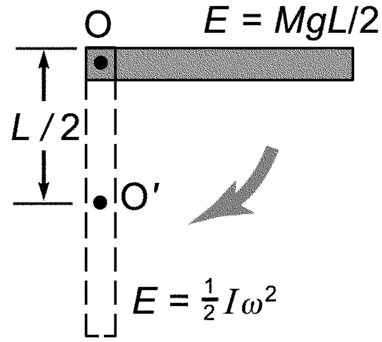
or $$\frac{1}{2}I\omega_1^2 + U_1 = \frac{1}{2}I\omega_2^2 + U_2$$

$\Rightarrow$ Cons. of Mechanical Energy in Rot. Motion

$$E = \frac{1}{2}I\omega^2 + U = \text{Constant}.$$

Example: Rotating Rod

Uniform rod of length L and mass M rotates about its end.
What is angular velocity at its lowest position?



Use Energy Conservation.
Assume zero of PE is at midpoint when rod is hanging down.

$$E_i = E_f$$

$$U_1 + \frac{1}{2}I\omega_1^2 = U_2 + \frac{1}{2}I\omega_2^2$$

$$\frac{1}{2}MgL + 0 = 0 + \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$

$$\omega = \sqrt{\frac{3g}{L}}$$

$L = 1\,m$
$\omega = 5.42\ r/s.$

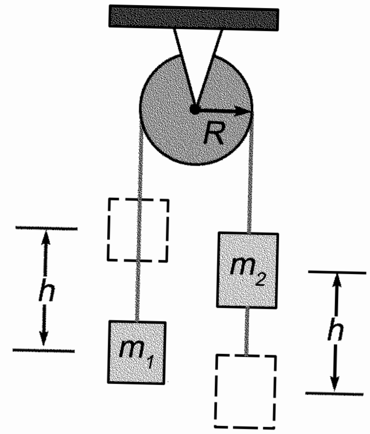
What is linear velocity?

Center: $v_c = r\omega = \frac{L}{2}\omega = \frac{1}{2}\sqrt{3gL}$

End : $v_e = r\omega = L\omega = \sqrt{3gL}$

Example: Connected Masses

Two masses connected by string passing over a polley of moment-of-inertia I.
What are linear velocities of masses after they move a height h?
No friction: Total energy is conserved.



$$E_1 = E_2$$

$$K_1 + U_1 = K_2 + U_2$$

$$0 = \frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 + \underbrace{\frac{1}{2}I\omega^2}_{\frac{1}{2}\frac{Iv^2}{R^2}} + m_1 g h - m_2 g h$$

$$v = R\omega$$

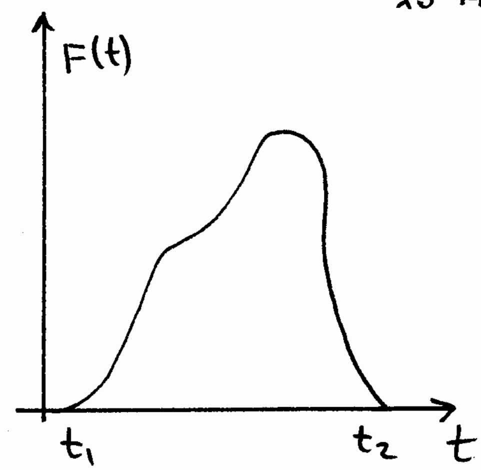
solve for $v = \left[\dfrac{2(m_2 - m_1)gh}{m_1 + m_2 + \frac{I}{R^2}}\right]^{1/2}$.

Alternate: $\tau = I\alpha$, solve for α and subsequent motion.

Angular Impulse

We showed previously that if we defined a linear impulse

$$\vec{I} = \int_{t_1}^{t_2} F(t)\, dt$$



The change in linear momentum

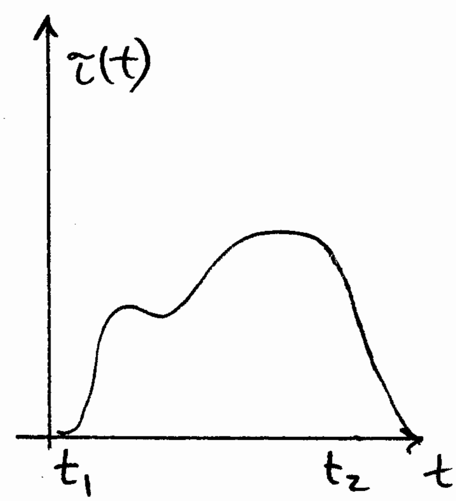
$$\frac{d\vec{p}}{dt} = \vec{F}$$

in a time interval $\Delta t = t_2 - t_1$

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{I}$$

[Area under F-t curve]

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$



$$\vec{L}_f - \vec{L}_i = \Delta\vec{L} = \vec{\tau}\,\Delta t$$

$$= \vec{J} \text{ (ang. impulse)}$$

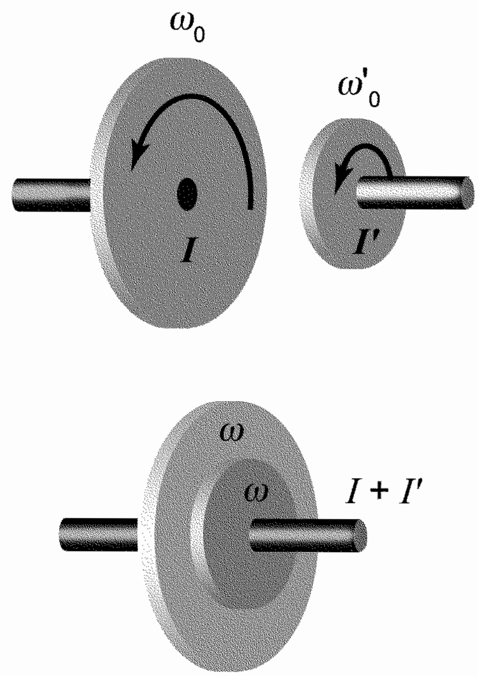
$$\vec{J} = \int_{t_1}^{t_2} \vec{\tau}(t)\, dt$$

applied torque as a function of time.

Change in $\vec{L}$ is equal to the angular impulse.

Example

Two disks with moments-of-inertia I and I' are rotating with velocities ω_0 and ω_0'.
Disks pushed together by an external force parallel to axis of rotation. No torques about this axis.
Disks rotate with final ang. velocity ω.



An impulsive torque acts when two rotating disks engage.

Disks exert torques on each other.
Corresponding impulses

$$J_\theta = -J_\theta'$$

$$J_\theta = I\omega - I\omega_0 \qquad \text{[Large disk]}$$

$$J_\theta' = I'\omega - I'\omega_0' \qquad \text{[Small disk]}$$

$$\therefore \quad I\omega - I\omega_0 = -(I'\omega - I'\omega_0') \qquad [J_\theta = -J_\theta']$$

$$\omega = \frac{I\omega_0 + I'\omega_0'}{I + I'}$$

Internal Torques: Total $\vec{L}$ is conserved.

Totally inelastic rotational collision!!
Kinetic Energy is not conserved.

$$L_i = I\omega_0 + I'\omega_0'$$

$$L_f = (I + I')\omega$$

$$L_i = L_f$$

$$I\omega_0 + I'\omega_0' = (I + I')\omega$$

$$\omega = \frac{I\omega_0 + I'\omega_0'}{I + I'}$$

Rotation plus Translation

Generalize idea of <u>fixed axis rotation</u>

i) Axis of rotation is fixed in an inertial frame

<u>or</u>

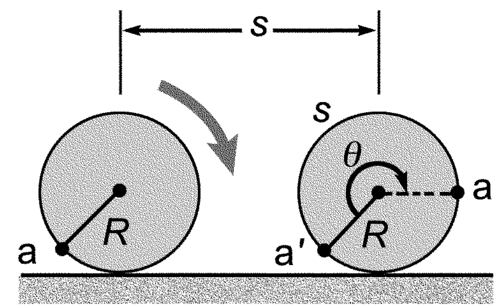
ii) Axis of rotation passes through the cm of the body and maintains a <u>constant direction</u> in space.

- in each case every point in body has a linear speed $v = r\omega$ due to rotation.
- in case ii) additional component due to translation of cm.
- rolling is a special motion of case ii)

Rolling Cylinders/Spheres

Uniform cylinder of radius R rolling on a flat surface. Cylinder rotates through angle θ, its cm moves a distance

$$s = R\theta$$

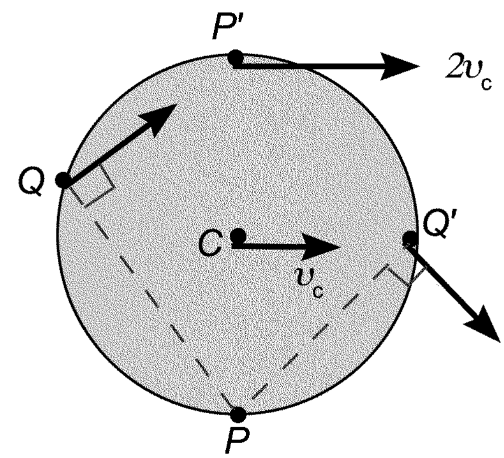


For pure rolling motion, as the cylinder rotates through an angle θ, the center of mass moves a distance s=Rθ.

The velocity and acceleration of the cm for pure rolling motion (no slipping) are given by:

$$v_c = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$a_c = \frac{dv_c}{dt} = R\frac{d\omega}{dt} = R\alpha$$



All points on a rolling body move in a direction perpendicular to an axis through the contact point P. The center of mass moves with a velocity v_c, while the point P' moves with the velocity $2v_c$.

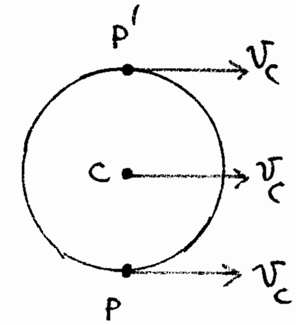
Different points on the object have different linear velocities. The linear velocity of any point is in a direction $\perp$ to the line from that point to the point of contact - P.

The contact point is at rest relative to the surface. An axis through P and $\perp$ $\vec{v}_c$ is the instantaneous axis of rotation.

A general point Q on the cylinder has both horizontal and vertical components of velocity.
Points P and P′ are of special interest.

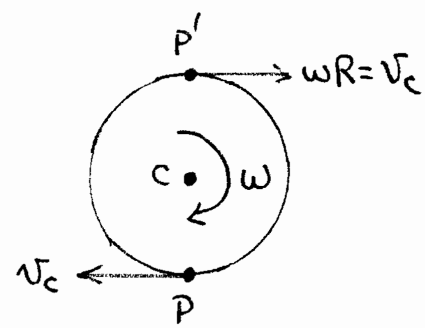
$$\left.\begin{array}{ll} P: & v_P = 0 \\ C: & v_C = R\omega \\ P': & v_{P'} = 2R\omega \end{array}\right\}$$ all points on cylinder have common ω.

<u>Translation</u>



- All points have the same speed $v = v_c$.

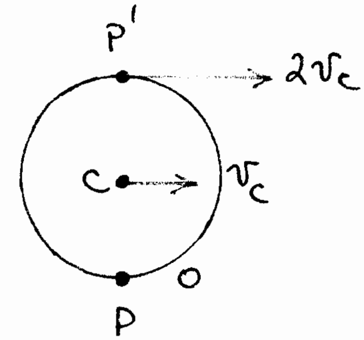
<u>Rotation</u>



$\omega R = v_c$

- All points have the same ang. velocity.
- Speed prop. to distance from axis.

<u>Translation + Rotation</u>



$$v_P = v_c - \omega R = 0$$

$$v_c = v_c + 0 = v_c$$

$$v_{P'} = v_c + \omega R = 2v_c$$

The combined effects of the translation of the cm and rotation about an axis through the cm are equivalent to a pure rotation with the same angular speed about an axis through the point of contact of the rolling body.

Kinetic Energy

Total KE is given by:

$$K = \frac{1}{2} I_P \omega^2$$

$$I_P = I_c + mR^2 \qquad [\text{Parallel Axis Thm}]$$

$$\therefore \quad K = \frac{1}{2} I_c \omega^2 + \frac{1}{2} mR^2 \omega^2$$

$$= \frac{1}{2} I_c \omega^2 + \frac{1}{2} m v_c^2 \qquad [\text{Consistent}]$$

Total KE for an object undergoing pure rolling motion is the sum of the rotational KE about the cm plus the translational KE of the cm.

Translation and Rotation

Theorem: Displacement of a rigid body can be decomposed into two independant motions: a translation of the CM and a rotation about the CM.

- Only motion for which the axis of rotation remains parallel to a given direction - say z-axis will be considered.

- Want to show that the total L_z can be written as the sum of two terms:

$$L_z = I_0 \omega + (\vec{R} \times M\vec{V})_z$$

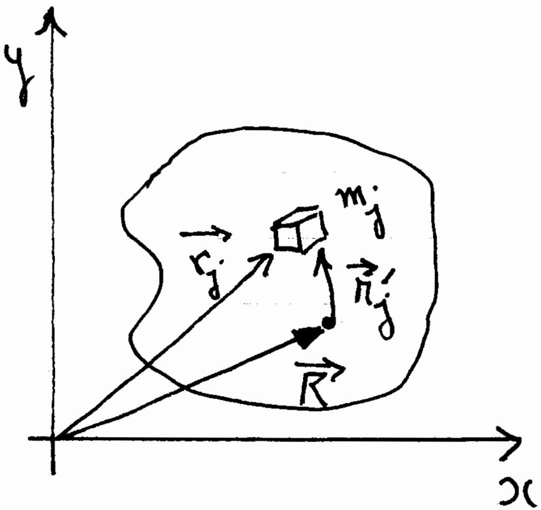
$\vec{R}$: Position vector of CM
$\vec{V}$: $= \dot{\vec{R}}$; velocity of CM

Consider body made up of m_j mass elements with position vectors $\vec{r}_j$.

$$\vec{L} = \sum_j \vec{r}_j \times m \dot{\vec{r}}_j$$

$$\vec{R} = \sum m_j \vec{r}_j$$

M = Total Mass



$$\vec{r}_j = \vec{R} + \vec{r}'_j$$

$$\vec{L} = \sum \vec{r}_j \times m_j \vec{v}_j$$

$$= \sum (\vec{R} + \vec{r}'_j) \times m_j (\dot{\vec{R}} + \dot{\vec{r}}'_j)$$

$$= \vec{R} \times \sum m_j \dot{\vec{R}} + \sum m_j \vec{r}'_j \times \dot{\vec{R}} + \vec{R} \times \sum m_j \dot{\vec{r}}'_j + \sum m_j \vec{r}'_j \times \dot{\vec{r}}'_j$$

Term-2: $\sum m_j \vec{r}'_j = \sum m_j (\vec{r}_j - \vec{R})$

$$= \sum m_j \vec{r}_j - M\vec{R}$$

$\equiv 0$ Def. of CM

Term-3: $\sum m_j \dot{\vec{r}}'_j \equiv 0$ Def. of CM Velocity in CM.

Term-1: $\vec{R} \times \sum m_j \dot{\vec{R}} = \vec{R} \times M\dot{\vec{R}}$

$$= \vec{R} \times M \vec{V}_{cm}$$

$\therefore$ $\vec{L} = \vec{R} \times M \vec{V}_{cm} + \sum \vec{r}'_j \times m_j \dot{\vec{r}}'_j$

First Term: $\vec{L_0}$: Angular momentum due to cm motion. Called orbital angular momentum.

Take a fixed axis for rotation: z-axis

$\Sigma(\vec{r_j'} \times m_j \vec{\dot{r}_j'})_z$: L_z, Spin Angular Momentum.

Body rotates about z-axis
For m_j; $|\vec{r_j}|$ a constant!

Picture

$\therefore \vec{v_j}$ must be $\perp$ to $\vec{r_j}$

$\perp$ distance of m_j to z-axis is ρ_j.

$$\therefore L_z(j) = m_j v_j \times \text{distance to axis}.$$
$$= m_j v_j \rho_j$$

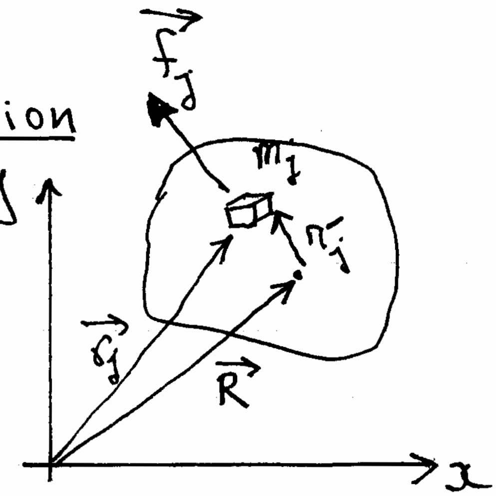
But $v_j = \omega \rho_j$ $\qquad$ ω: angular velocity.

$$\therefore L_z(j) = m_j \rho_j^2 \omega$$

$$L_z = \Sigma L_z(j) = \Sigma m_j \rho_j^2 \omega$$

$$L_z = I_z \omega$$ Spin Angular Momentum.

Torques: Rotation and Translation



$$\vec{\tau} = \sum \vec{r}_j \times \vec{f}_j$$
$$= \sum(\vec{r}_j{}' + \vec{R}) \times \vec{f}_j$$
$$= \sum(\vec{r}_j{}' \times \vec{f}_j) + (\vec{R} \times \vec{F})$$

Where, $\vec{F} = \sum f_j$; total applied force.

Consider rotation about fixed z-axis.

$$\vec{\tau}_z = \tau_0 + (\vec{R} \times \vec{F})_z$$

τ_0 = z-component of torque about cm

We had : $L_z = I_0\omega + (\vec{R} \times M\vec{V})_z$

$$\frac{dL_z}{dt} = I_0\frac{d\omega}{dt} + \frac{d}{dt}(\vec{R} \times M\vec{V})_z$$
$$= I_0\alpha + (\vec{R} \times M\vec{a})_z$$

$$\tau_z = dL_z/dt$$

$$\therefore \tau_0 + (\vec{R} \times \vec{F})_z = I_0\alpha + (\vec{R} \times M\vec{a})_z$$
$$= I_0\alpha + (\vec{R} \times \vec{F})_z \qquad \vec{F} = M\vec{a}$$

$$\therefore \tau_0 = I_0\alpha$$

$\therefore$ Rotational motion about cm depends only on torque about cm, independant of the translational motion. Result is correct even if the axis of rotation is accelerating.