Solutions

BLOCK 3:
PARTIAL DERIVATIVES

Solutions
Block 3: Partial Derivatives

1. Use polar coordinates to obtain $w = \dfrac{2r \sin \theta \ (r \cos \theta)}{r^2} =$

   $2 \sin \theta \cos \theta$ if $r \neq 0$. Therefore, $w = \sin 2\theta$ $(r \neq 0)$ and
   therefore, $\displaystyle\lim_{(x,y) \to (0,0)} w = \lim_{(r,\theta) \to (0,\theta_o)} w = \sin 2\theta_o$ and this depends
   on θ_o.

2. $5x + 7y + 6z = 18$.

3. $h''(r) + \dfrac{1}{r} h'(r)$.

4. $x^2 e^y + e^x + e^y = 2$.

5. $\dfrac{1}{y + 1}$.

Unit 1: Functions of More Than One Variable

___3.1.1(L)___

While it was more natural, in terms of an extension of lower
dimensional cases, to invent the Euclidean metric, the point is
that a metric is defined solely by the properties mentioned in
this exercise. Since, from a logical point of view, we use nothing
but inescapable consequences of these properties, it follows
that any metric that has these same properties may be used in our
study. In particular, we want to show in this exercise that the
Minkowski metric has these properties. Thus,

a. Since $\|\underline{x}\| = \max\{|x_1|, \ldots, |x_n|\}$ the fact that each $|x_i|$, $(i=1,\ldots,n)$,
   is at least as great as zero guarantees that the maximum of the
   x_i's is also at least as great as zero. This proves that

$$\|\underline{x}\| \geqslant 0$$

Now, if it happens that $\|\underline{x}\|$ actually equals 0, then by definition,
the maximum $|x_i|$ is also zero. But no $|x_i|$ can be less than zero;
thus, the maximum $|x_i|$ being zero guarantees that all $|x_i|$ equal
zero or that all $x_i = 0$, and this means that

$$\underline{x} = (x_1, \ldots, x_n) = (0, \ldots, 0) = \underline{0}.$$

As for checking that $\|\underline{x}+\underline{y}\| \leqslant \|\underline{x}\| + \|\underline{y}\|$, we see from the definition
of the Minkowski metric that this is equivalent to proving that

$$\max\{|x_1+y_1|, \ldots, |x_n+y_n|\} \leqslant \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\}$$

$$\tag{1}$$

Unless we are bothered by the fairly abstract notation, (1) is
almost trivial to prove. Namely, let $|x_k+y_k|$ denote the maximum
of $|x_1+y_1|, \ldots,$ and $|x_n+y_n|$. We then have

$$\max\{|x_1+y_1|, \ldots, |x_n+y_n|\} = |x_k+y_k| \quad ;$$

3.1.1(L) continued

and, since x_k and y_k are real numbers, we already know from part 1 of our course that $|x_k+y_k| \leq |x_k| + |y_k|$. Hence,

$$\max\{|x_1+y_1|,\ldots,|x_n+y_n|\} \leq |x_k| + |y_k| \tag{2}$$

Finally, since $|x_k| \leq \max\{|x_1|,\ldots,|x_n|\}$ and $|y_k| \leq \max\{|y_1|,\ldots,|y_n|\}$, we have from (2) that

$$\max\{|x_1+y_1|,\ldots,|x_n+y_n|\} \leq \max\{|x_1|,\ldots,|x_n|\}+ \max\{|y_1|,\ldots|y_n|\}.$$

In fact, if the x's and y's all have the same sign, equality can hold if and only if

$$|x_k| = \max\{|x_1|,\ldots,|x_n|\} \quad \underline{and} \quad |y_k| = \max\{|y_1|,\ldots,|y_n|\} ,$$

which means that the maximum components of $\underline{x}$ and $\underline{y}$ "match" (i.e., have the same subscript).

For example, if $\underline{x} = (2,4,1)$ and $\underline{y} = (3,2,5)$, the maximum components do not match (in $\underline{x}$ it is the second component, 4, and in $\underline{y}$ it is the third component, 5). In this case $\underline{x}+\underline{y} = (5,6,6)$, whence $\|\underline{x}+\underline{y}\| = 6$; while $\|\underline{x}\| + \|\underline{y}\| = 4+5 = 9$. Thus $\|\underline{x}+\underline{y}\| < \|\underline{x}\| + \|\underline{y}\|$.

On the other hand if $\underline{x} = (2,5,1)$ and $\underline{Y} = (3,7,2)$ [where the maximum components match], $\|\underline{x}+\underline{y}\| = \max\{5,12,3\} = 12$ while $\|\underline{x}\| + \|\underline{y}\| = 5+7 = 12$, so that $\|\underline{x}+\underline{y}\| = \|\underline{x}\| + \|\underline{y}\|$.

Finally, we must show that

$$\|a\ \underline{x}\| = |a|\ \|\underline{x}\| \quad .$$

At any rate,

3.1.1(L) continued

$$\|a \ \underline{x}\| = \max\{|ax_1|,\ldots,|ax_n|\}$$

$$= \max\{|a||x_1|,\ldots,|a||x_n|\}$$

$$= |a| \ \max\{|x_1|,\ldots,|x_n|\}^*$$

$$= |a| \ \|\underline{x}\|$$

b. From a mechanical point of view, this is hardly a learning exercise.
    In fact, it is more like a trivial drill exercise. We are given
    that $\underline{x} = (2,4,1)$ while $\underline{y} = (4,4,5)$. (We picked n = 3 so that you
    could use your geometric intuition if you so desired; that is, with
    n = 3 we may think of $\underline{x}$ as being $2\vec{i} + 4\vec{j} + \vec{k}$, etc).
    Then,

$$\underline{x} \cdot \underline{y} = (2)(4) + (4)(4) + (1)(5) = 29 \tag{1}$$

On the other hand, our definition of the Minkowski metric yields
that

$$\|\underline{x}\| = \max\{2,4,1\} = 4 \tag{2}$$

and

$$\|\underline{y}\| = \max\{4,4,5\} = 5 \tag{3}$$

From (2) and (3) we have that $\|\underline{x}\| \ \|\underline{y}\| = 20$, and comparing this
result with (1), we have that

$$|\underline{x} \cdot \underline{y}| > \|\underline{x}\| \ \|\underline{y}\| \tag{4}$$

---

*Notice that if c>0 is any constant, then

$$\max\{cx_1,\ldots cx_n\} = c \max\{x_1,\ldots,x_n\}$$

since c "magnifies" each x_i in the same ratio.

3.1.1(L) continued

Equation (4) violates a key property of a dot product, especially as we know it in the 2- and 3-dimensional cases. In other words, the Minkowski metric does not obey the structural property that is so often used in dot products (namely, $|\underline{x} \cdot \underline{y}| \leqslant \|\underline{x}\| \ \|\underline{y}\|$) while the Euclidean metric does have this structural property (as we have proven in our supplementary notes).

This explains why most texts stress the Euclidean metric over the Minkowski metric, even though as we shall see in the next few exercises, the Minkowski metric is most helpful in our discussion of limits in the case of real functions of n-tuples, expecially with n>3.

In our later work (especially Block 7) we will do a great deal of work with the generalized notion of a dot product and in this context the Euclidean metric has desired properties not shared by the Minkowski metric.

From the point of view of our "game," however, the Minkowski metric is on a par with the Euclidean metric in the type of investigation we are currently undertaking where the dot product of two n-tuples is not an issue.

3.1.2

We have $\displaystyle\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = L_1$ and $\displaystyle\lim_{\underline{x}\to\underline{a}} g(\underline{x}) = L_2$.

Given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$0 < \|\underline{x}-\underline{a}\| < \delta \rightarrow |[f(\underline{x})+g(\underline{x})] - [L_1+L_2]| < \varepsilon.$$

Just as in the scalar case, we observe that

$$|[f(\underline{x})+g(\underline{x})] - [L_1+L_2]| = |[f(\underline{x})-L_1] + [g(\underline{x})-L_2]|$$

$$\leqslant |f(\underline{x})-L_1| + |g(\underline{x})-L_2| \qquad (1)$$

By definition of L_1 and L_2, we can find δ_1 and δ_2 such that

3.1.2 continued

$$0 < \|\underline{x}-\underline{a}\| < \delta_1 \rightarrow |f(\underline{x})-L_1| < \frac{\epsilon}{2}$$

$$0 < \|\underline{x}-\underline{a}\| < \delta_2 \rightarrow |g(\underline{x})-L_2| < \frac{\epsilon}{2}$$

(2)

Thus if $\delta \leq \min\{\delta_1, \delta_2\}$ it follows from (2) that

$$0 < \|\underline{x}-\underline{a}\| < \delta \rightarrow |f(\underline{x})-L_1| + |g(\underline{x})-L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad (3)$$

Combining (3) with (1) yields the desired result, namely

$$0 < \|\underline{x}-\underline{a}\| < \delta \rightarrow |[f(\underline{x})+g(\underline{x})] - [L_1+L_2]| < \epsilon$$

Note:

In this exercise, we did not have to specify whether it was the Euclidean metric or the Minkowski metric that was being used. The point is that our proof used only those properties that were true for either metric. To be sure, the actual value of δ for a given value of ϵ might depend on which metric was used, but the existence of δ did not.

On the other hand, there may be a tendency to think in terms of the Minkowski metric when we say "let $\delta = \min\{\delta_1, \delta_2\}$." The point is that while this part of our logical strategy is the same whether it is the Euclidean metric or the Minkowski metric which is being used, what is true is that the motivation behind the strategy is the same as that which inspired the "invention" of the Minkowski metric. Namely, in both instances, the idea is that to make sure that something is sufficiently small, we make sure that its maximum dimension is sufficiently small, regardless of how we define "dimension."

One final word that may be of interest is that in our 1-dimensional study of calculus, that is, when we were studying limits of functions of a single real variable, there was no distinction between the Euclidean and the Minkowski metric. The interesting point is that

in the special case of n=1, the two metrics are identical. Namely, for a real number, x, $|x|$ and $\sqrt{x^2}$ are the same (and, obviously, for the case of only one component, $|x| = \max\{|x|\}$). That is, either the Minkowski or the Euclidean metric would have led to the same definition of magnitude in the case of a 1-dimensional space.

3.1.3(L)

a. We must find δ such that

$$0 < \|(x,y)-(2,3)\| < \delta \rightarrow |x^2+y^3-31| < \varepsilon$$

As seen in our supplementary notes, $|x^2+y^3-31| < \varepsilon$ if

$$|x^2-4| < \frac{\varepsilon}{2} \quad \text{and} \quad |y^3-27| < \frac{\varepsilon}{2}$$

To make $|x^2-4| < \frac{\varepsilon}{2}$ we take into consideration that x is near 2 (i.e., ε is small). That is, we may assume that

$$2-\xi < x < 2 + \xi \quad \text{where } |\xi| < 1 \quad .$$

In particular, then,

$$1 < x < 3$$

and therefore

$$3 < x+2 < 5 \tag{1}$$

Thus, near x=2,

$$|x^2-4| = |x-2||x+2| = |x-2|(x+2)$$

or, from (1)

$$|x^2-4| < 5|x-2| \tag{2}$$

From (2) it is easily seen that $|x-2| < \frac{\varepsilon}{10}$ implies that $|x^2-4| < \frac{\varepsilon}{2}$.

3.1.3(L) continued

Thus, the δ_1 referred to in our notes may be taken to be $\frac{\varepsilon}{10}$. In other words,

$$0 < |x-2| < \frac{\varepsilon}{10} \rightarrow |x^2-4| < \frac{\varepsilon}{2} \tag{3}$$

As for making $|y^3-27| < \frac{\varepsilon}{2}$, we observe that

$$y^3-27 = (y-3)(y^2+3y+9) \quad .$$

Hence

$$|y^3-27| = |y-3||y^2+3y+9| \tag{4}$$

Since we know that y is near 3, it follows that y^2+3y+9 is near 27 (because we already know that y^2+3y+9 is a continuous function of y). In particular, we shall assume y is close enough to 3 so that y^2+3y+9 $(=|y^2+3y+9|) < 28.$

With this in mind, (4) becomes

$$|y^3-27| = |y^2+3y+9||y-3|$$
$$< 28|y-3| \quad . \tag{5}$$

From (5), it follows immediately that if $|y-3| < \frac{\varepsilon}{56}$ then $|y^3-27| < \frac{\varepsilon}{2}.$ In other words

$$0 < |y-3| < \frac{\varepsilon}{56} \rightarrow |y^3-27| < \frac{\varepsilon}{2} \quad ,$$

and $\frac{\varepsilon}{56}$ is the δ_2 of our notes.

Since ε is positive, it is clear that $\frac{\varepsilon}{56} < \frac{\varepsilon}{10}$. Hence, if $\delta = \min\{\delta_1, \delta_2\}$ we have

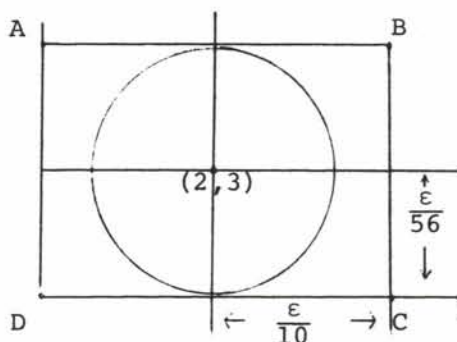$$\underline{\delta = \frac{\varepsilon}{56}} \quad .$$

3.1.3(L) continued

In summary, using the Minkowski metric, given $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{56}$.
Then

$$0 < \|(x,y) - (2,3)\| < \delta \rightarrow |x^2 + y^3 - 31| < \varepsilon$$

[and by our remarks in the supplementary notes, the value of δ also
suffices if we are using the Euclidean metric, as we shall review
in part (b)].

b. Pictorially, we have

Since $|x^2 + y^3 - 31|$ is true for all (x,y) in rectangle ABCD, it is, in
particular,   true for all (x,y) in the circle centered at $(2,3)$
with radius $\frac{\varepsilon}{56}$. The equation of this circle is

$$(x-2)^2 + (y-3)^2 = \left(\frac{\varepsilon}{56}\right)^2 .$$

In any event

$$\sqrt{(x-2)^2 + (y-3)^2} < \frac{\varepsilon}{56} \rightarrow |x^2 + y^3 - 31| < \varepsilon$$

3.1.3(L) continued

Hence,

$$0 < \|(x,y)-(2,3)\| < \frac{\varepsilon}{56} \rightarrow |x^2+y^3-31| < \varepsilon$$

is true even if $\|(x,y)-(2,3)\|$ now denotes the Euclidean metric.

___3.1.4(L)___

We have $\underline{x} = (x_1,x_2,x_3,x_4)$, $\underline{1} = (1,1,1,1)$ and $f(\underline{x})=x_1^2+2x_2+x_3^3+x_4^2$.
We must prove that

$$\lim_{\underline{x}\to\underline{1}} f(\underline{x}) = f(\underline{1}) \tag{1}$$

Since $\underline{1} = (1,1,1,1)$, the definition of f tells us that
$f(\underline{1}) = f(1,1,1,1) = 1^2+2(1) + 1^3 + 1^2 = 5$. Thus, we see that
$f(\underline{x})$ exists at $\underline{1}$ and its value is 5. Next we must show that

$$\lim_{\underline{x}\to\underline{1}} (x_1^2+2x_2+x_3^3+x_4^2) = 5 \tag{2}$$

Equation (2), by definition, now implies that for a given $\varepsilon>0$ we
must find $\delta>0$ such that

$$0 < \|\underline{x}-\underline{1}\| < \delta \rightarrow |x_1^2+2x_2+x_3^3+x_4^2-5| < \varepsilon \tag{3}$$

Up to now we have not specified what metric we are using. If we
specify the Minkowski metric (noting that the same δ will also
work for the Euclidean metric), (3) becomes

Find $\delta>0$ such that

$$\max\{|x_1-1|, |x_2-1|, |x_3-1|, |x_4-1|\} < \delta \rightarrow$$

$$|x_1^2+2x_2+x_3^3+x_4^2-5| < \varepsilon \tag{4}$$

3.1.4(L) continued

We rewrite $|x_1^2+2x_2+x_3^3+x_4^2-5|$ in the more suggestive form

$$|(x_1^2-1) + 2(x_2-1) + (x_3^3-1) + (x_4^2-1)|$$

(since this is how we got 5 in the first place), where upon we may conclude that

$$|x_1^2+2x_2+x_3^3+x_4^2-5| \leq |x_1^2-1|+2|x_2-1|+|x_3^3-1|+|x_4^2-1| \qquad (5)$$

From (5), $|x_1^2+2x_2+x_3^3+x_4^2-5|$ will be less than ε as soon as

$$|x_1^2-1|+2|x_2-1|+|x_3^3-1|+|x_4^2-1| < \varepsilon.$$

This, in turn, will happen as soon as each of our four summands is less than $\frac{\varepsilon}{4}$. That is, we need only make each of the numbers $|x_1^2-1|$, $2|x_2-1|$, $|x_3^3-1|$, and $|x_4^2-1|$ less than $\frac{\varepsilon}{4}$. In other words we must have:

(1) $\quad |x_1^2-1| < \frac{\varepsilon}{4}$

(2) $\quad 2|x_2-1| < \frac{\varepsilon}{4}$ (or $|x_2-1| < \frac{\varepsilon}{8}$)

(3) $\quad |x_3^3-1| < \frac{\varepsilon}{4}$ $\qquad\qquad\qquad\qquad$ (6)

(4) $\quad |x_4^2-1| < \frac{\varepsilon}{4}$

The key now is that each of the four inequalities in (6) involves only scalars. In particular, since we know that

$$\lim_{x_1 \to 1} x_1^2 = 1$$

$$\lim_{x_2 \to 1} x_2 = 1$$

$$\lim_{x_3 \to 1} x_3^3 = 1$$

$$\lim_{x_4 \to 1} x_4^2 = 1 \qquad\qquad ,$$

3.1.4(L) continued

it follows that we can find δ_1, δ_2, δ_3, and δ_4 such that

$$0 < |x_1-1| < \delta_1 \rightarrow |x_1{}^2-1| < \frac{\varepsilon}{4}$$

$$0 < |x_2-1| < \delta_2 \rightarrow |x_2-1| < \frac{\varepsilon}{8}$$

$$0 < |x_3-1| < \delta_3 \rightarrow |x_3{}^3-1| < \frac{\varepsilon}{4}$$ 　　(7)

$$0 < |x_4-1| < \delta_4 \rightarrow |x_4{}^2-1| < \frac{\varepsilon}{4}$$

Hence, if we let $\delta = \min\{\delta_1,\delta_2,\delta_3,\delta_4\}$, we may conclude from (7) that if $|x_i=1|<\delta$ (i=1,2,3,4) then

$$|x_1{}^2-1| + 2|x_2-1| + |x_3{}^3-1| + |x_4{}^2-1| < \varepsilon$$

and from (5) this guarantees that

$$|x_1{}^2+2x_2+x_3{}^3+x_4{}^2-5| < \varepsilon.$$

In other words, from the definition of the Minkowski metric,

$$0 < \|\underline{x}-\underline{1}\| < \delta \rightarrow \max\{|x_1-1|, |x_2-1|, |x_3-1|, |x_4-1|\} < \delta \rightarrow$$

$$|x_i-1| < \delta, \ i = 1,2,3,4 \rightarrow$$

$$|x_1{}^2+2x_2+x_3{}^3+x_4{}^2-5| < \varepsilon$$

which was precisely what had to be shown.

A major aim of this exercise is to have you become convinced that the idea of a metric applies to higher dimensional cases (in our example, we used it in the 4-dimensional case), and that the Minkowski metric allows us to find an n-dimensional distance in terms of n 1-dimensional distances.

3.1.5

Given $\varepsilon > 0$ we wish to determine δ, in terms of ε such that

$$\|\underline{x}-\underline{1}\| < \delta \to |x_1{}^2+2x_2+x_3{}^3+x_4{}^2-5| < \varepsilon .$$

We may take equation (6) of the previous exercise as our starting point.

Namely, near $x_1 = 1$, $x_1 + 1$ is near 2. In particular, we may assume $|x_1+1| = x_1+1 < 3$. Then

$$|x_1{}^2-1| = |x_1+1||x_1-1| < 3 |x_1-1|$$

Hence, if we make sure that $|x_1-1| < \frac{\varepsilon}{12}$, we obtain

$$|x_1{}^2-1| < \frac{3\varepsilon}{12} = \frac{\varepsilon}{4} .$$

In summary, with $\underline{\delta_1 = \frac{\varepsilon}{12}}$, we have

$$0 < |x_1-1| < \delta_1 \to |x_1{}^2-1| < \frac{\varepsilon}{4}$$

and (1) of Equation (6) in the previous exercise is obeyed.

Next, with respect to (2) of Equation (6), we want $|x_2-1| < \frac{\varepsilon}{8}$, but now it is trivial to see that $\underline{\delta_2 = \frac{\varepsilon}{8}}$, allows us to say

$$0 < |x_2-1| < \delta_2 \to |x_2-1| < \frac{\varepsilon}{8} .$$

As for (3) of Equation (6), we first write

$$x_3{}^3-1 = (x_3-1)(x_3{}^2+x_3+1)$$

when x_3 is near 1, $x_3{}^2+x_3+1$ is near 3. Again, we pick our interval

small enough so that $|x_3{}^2+x_3+1| = x_3{}^2+x_3+1 < 4$. Once this is done we have

$$|x_3{}^3-1| = |x_3-1||x_3{}^2+x_3+1| < 4|x_3-1|$$

3.1.5 continued

and if we now make sure that $|x_3-1| < \frac{\varepsilon}{16}$, we have
$|x_3{}^3-1| < 4 (\frac{\varepsilon}{16}) = \frac{\varepsilon}{4}$, and (3) is obeyed.

In summary, if we let $\underline{\delta_3 = \frac{\varepsilon}{16}}$ then

$$0 < |x_3-1| < \delta_3 \rightarrow |x_3{}^3-1| < \frac{\varepsilon}{4}$$

Finally, we treat (4) of Equation (6) by observing that near 1,
x_4+1 is near 2, hence, for a sufficiently small neighborhood of
1, we may conclude that $|x_4+1| = x_4+1 < 3$. Then,

$$|x_4{}^2-1| = |x_4+1||x_4-1| < 3|x_4-1|$$

so that if $|x_4-1| < \frac{\varepsilon}{12}$, then $|x_4{}^2-1| < \frac{\varepsilon}{4}$. That is, in this
case we may let $\underline{\delta_4 = \frac{\varepsilon}{12}}$.

Letting $\delta = \min\{\delta_1,\delta_2,\delta_3,\delta_4\}$, we have $\delta = \{\frac{\varepsilon}{12}, \frac{\varepsilon}{8}, \frac{\varepsilon}{16}, \frac{\varepsilon}{12}\}$, and
since $\varepsilon>0$, the fraction with the greatest denominator will be
the minimum.

Hence, we may let

$$\delta = \frac{\varepsilon}{16} \quad .$$

That is

$$0 < \|(x_1,x_2,x_3,x_4)-(1,1,1,1)\| < \frac{\varepsilon}{16} \rightarrow |x_1{}^2+2x_2+x_3{}^3+x_4{}^2-5| < \varepsilon$$

The key point is that δ was found by looking at four 1-dimensional
limits, and it is in this sense that higher dimensional problems
are as real as the lower dimensional ones. Moreover, for every
$\underline{x}$ such that $\|\underline{x}-\underline{1}\| < \frac{\varepsilon}{16}$, $|f(\underline{x})-5| < \varepsilon$.

(Note: If you have no quantitative feeling for $\delta = \frac{\varepsilon}{16}$ in this
exercise, it might be worthwhile to carry the computational details
a bit further. Namely, $\delta = \frac{\varepsilon}{16}$ describes the domain

3.1.5 continued

$$
\left.\begin{array}{l}
1 - \dfrac{\varepsilon}{16} < x_1 < 1 + \dfrac{\varepsilon}{16} \\[2mm]
1 - \dfrac{\varepsilon}{16} < x_2 < 1 + \dfrac{\varepsilon}{16} \\[2mm]
1 - \dfrac{\varepsilon}{16} < x_3 < 1 + \dfrac{\varepsilon}{16} \\[2mm]
1 - \dfrac{\varepsilon}{16} < x_4 < 1 + \dfrac{\varepsilon}{16}
\end{array}\right\} \quad \text{i.e., a "4-dimensional cube"}
$$

Assuming that $\varepsilon > 0$ is sufficiently small to insure that all numbers in the above inequalities are positive, we have

$$
1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} < x_1^2 < 1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256}
$$

$$
2 - \frac{\varepsilon}{8} < 2x_2 < 2 + \frac{\varepsilon}{8}
$$

$$
1 - \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} - \frac{\varepsilon^3}{4096} = \left(1 - \frac{\varepsilon}{16}\right)^3 < x_3^3 < \left(1 + \frac{\varepsilon}{16}\right)^3
$$

$$
= 1 + \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} + \frac{\varepsilon^3}{4096}
$$

$$
1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} < x_4^2 < 1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256}
$$

Therefore

$$
\left(1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} + 2 - \frac{\varepsilon}{8} + 1 - \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} - \frac{\varepsilon^3}{4096} + 1 - \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} \right)
$$

is a lower bound for

$$
x_1^2 + 2x_2 + x_3^3 + x_4^2
$$

in this case, while

$$
\left(1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} + 2 + \frac{\varepsilon}{8} + 1 + \frac{3\varepsilon}{16} + \frac{9\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} + 1 + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{256} \right)
$$

is an upper bound.

3.1.5 continued

Simplifying we find that $x_1{}^2 + 2x_2 + x_3{}^3 + x_4{}^2$ is between

$$5 - \frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} - \frac{\varepsilon^3}{4096} \tag{1}$$

and

$$5 + \frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} \tag{2}$$

Our claim now is that for ε sufficiently small the numbers named by (1) and (2) be between $5-\varepsilon$ and $5+\varepsilon$. For example, assuming that $\varepsilon < 1$, we have $\varepsilon < \varepsilon^2 < \varepsilon^3$, whereupon

$$\frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} < \frac{9\varepsilon}{16} + \frac{11\varepsilon}{256} + \frac{\varepsilon}{4096} = \frac{\varepsilon}{4096}[9(256) + 11(16) + 1]$$

Therefore,

$$\frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} < \frac{\varepsilon}{4096}(2304 + 176 + 1) < \varepsilon \tag{3}$$

[Actually, by choosing δ to be the $\underline{\text{minimum}}$ of δ_1, δ_2, δ_3, and δ_4 we narrowed the interval and this is why $\frac{2481}{4096}\varepsilon$ occurred rather than something closer in value to ε].

In any event, (3) shows that

$$5 + \frac{9\varepsilon}{16} + \frac{11\varepsilon^2}{256} + \frac{\varepsilon^3}{4096} < 5 + \varepsilon).$$

3.1.6(L)

a. Many of the same questions that could have been asked about limits in our study of calculus of a single variable can also be asked now. Among other things, the form 0/0 enters into things here just as before. Part (a) of this exercise is meant to emphasize that $f(x,y)$ cannot be continuous at (0,0) since it is not even defined there.
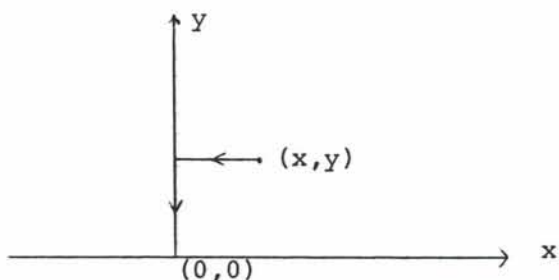
3.1.6(L) continued

In still other words, if f were continuous at (0,0), it would mean that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

but this is impossible since f(0,0) is not defined (0/0).

b. Even though f is not defined at (0,0), it does not prevent our writing $\lim_{(x,y)\to(0,0)} f(x,y)$ since then we are interested in what happens to f as (x,y) gets arbitrarily close to (0,0) <u>without equaling</u> (0,0). The problem is that the limit might (and in this case, it does) depend on just how (x,y) approaches (0,0). For example, suppose both x and y are not equal to 0 and we let (x,y) approach (0,0) along the indicated path:

(Figure 1)

In terms of the indicated path, we first hold y constant and let x approach 0. As soon as our path reaches the y-axis we hold x constant (x=0) and let y approach 0. Computing the limit this way means

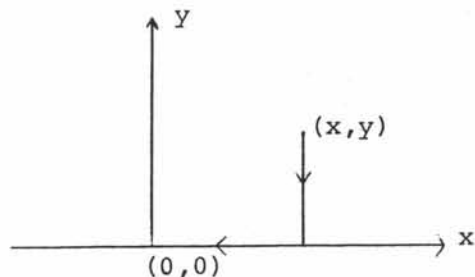$$\lim_{y\to0} \left[\lim_{x\to0} f(x,y) \right] \tag{1}$$

3.1.6(L) continued

Notice that (1) is a special way in which we may write

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

That is, (1) indicates but <u>one</u> of the infinitely many paths by which (x,y) may approach $(0,0)$. Evaluating (1) yields

$$\lim_{y\to0}\left[\lim_{x\to0}\frac{x^2-y^2}{x^2+y^2}\right] = \lim_{y\to0}(-1) = -1 \qquad (2)$$

On the other hand, the path

suggests

$$\lim_{x\to0}\left[\lim_{y\to0}\frac{x^2-y^2}{x^2+y^2}\right] = \lim_{x\to0}\frac{x^2}{x^2} = 1 \qquad (3)$$

A comparison of (2) and (3) shows that the value of

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

in this example depends on the path by which we approach $(0,0)$; since even for the two special paths we considered, we got two different answers.

The point is that when we say

$$\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = L$$

3.1.6(L) continued

we mean the limit exists and is L <u>for every possible path</u> by which $\underline{x} \to \underline{a}$. (In higher dimensions it means that when we make $|x_1-a_1|,\ldots,$ and $|x_n-a_n|$ small, our numerical answer must be independent of this order in which we make these quantities small, and this, in turn, says that our answer is uniquely determined once x_k is sufficiently close to a_k for $k = 1,\ldots,n$).

From a more positive point of view, if we are told that

$$\lim_{(x,y) \to (a,b)} f(x,y)$$

exists, then we can be sure that its value, among other ways, is given by either

$$\lim_{y \to b} \left[\lim_{x \to a} f(x,y) \right] \quad \text{or} \quad \lim_{x \to a} \left[\lim_{y \to b} f(x,y) \right].$$

The main caution is that <u>unless</u> we know the limit exists, these two different limits <u>may</u> yield different answers, as in the case of the present exercise.

c. Here we try to show what goes wrong near the origin by utilizing polar coordinates. (Among other things, this shows us that polar coordinates have a purpose quite apart from central force fields!)

Letting $x = r \cos \theta$ and $y = r \sin \theta$ we see that

$$\frac{x^2-y^2}{x^2+y^2} = \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2}$$

$$= \cos^2\theta - \sin^2\theta \quad , \quad \text{provided } r \neq 0$$

$$= \cos 2\theta \quad , \quad \text{provided } r \neq 0$$

3.1.6(L) continued

In other words, computing

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

is equivalent to computing $\lim_{r\to 0} \cos 2\theta$.

The point is that the value of $\cos 2\theta$ depands only on θ - not r. For example, if we choose the ray $\theta = \frac{\pi}{6}$, then $\cos 2\theta = \frac{1}{2}$. Therefore, if $(x,y)\to(0,0)$ along the ray $\theta = \frac{\pi}{6}$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2} \quad .$$

As a check, notice that in Cartesian coordinates the ray $\theta = \frac{\pi}{6}$ becomes the half-line $\frac{y}{x} = \tan \frac{\pi}{6}$ in the first quadrant. Since $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ we have $y = \frac{x}{\sqrt{3}}$ on this ray. Thus,

$$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2-\frac{x^2}{3}}{x^2+\frac{x^2}{3}} = \frac{\frac{2}{3}x^2}{\frac{4}{3}x^2} = \frac{1}{2} \quad \text{if } x \neq 0$$

Therefore,
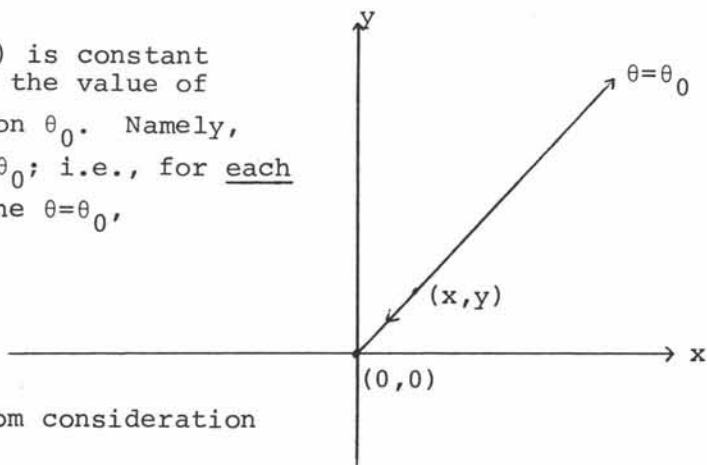
$$\lim_{(x,y)\to(0,0)} \left[\frac{x^2-y^2}{x^2+y^2} \right] = \frac{1}{2} \quad , \text{ as predicted.}$$

Pictorially,

3.1.6(L) continued

If $(x,y) \neq (0,0)$, $f(x,y)$ is constant
on each ray $\theta = \theta_0$, but the value of
the constant depends on θ_0. Namely,
the constant is $\cos 2\theta_0$; i.e., for <u>each</u>
point (x,y) on the line $\theta = \theta_0$,
$f(x,y) = \cos 2\theta_0$.

[$(0,0)$ is excluded from consideration
on $\theta = \theta_0$, since
$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{only if } (x,y) \neq (0,0)].$$

(Figure 2)

In summary, here, we have a rather exceptional example in which
we can make $f(x,y)$ take on any value in $[-1,1]$ merely by choosing
the appropriate path by which we allow $(x,y) \rightarrow (0,0)$.

3.1.7

a. We have

$$f(x,y) = \frac{2xy}{x^2 + y^2} \tag{1}$$

Letting $x = r \cos \theta$ and $y = r \sin \theta$, we have that

$$f(x,y) = f(r \cos \theta, r \sin \theta) = \frac{2r \cos \theta \, (r \sin \theta)}{r^2}$$

$$= 2 \sin \theta \cos \theta, \text{ provided } r \neq 0$$

therefore

$$f(x,y) = \sin 2\theta, \quad \text{provided } r \neq 0$$

3.1.7 continued

therefore

$$\lim_{\substack{(x,y)\to(0,0)}} f(x,y) = \lim_{\substack{r\to 0 \\ \theta=\theta_0}} \sin 2\theta = \sin 2\theta_0 \qquad (2)$$

Since the path is determined by θ_0, (2) shows that different paths
lead to different values for the limit. This will be explored
in more detail in the remaining parts of this exercise and also
in the next exercise.

b. If $\theta_0 = \frac{\pi}{4}$, (2) yields

$$\lim_{\substack{(x,y)\to(0,0)}} f(x,y) = \sin 2\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} = 1 . \qquad (3)$$

As a check $\theta_0 = \frac{\pi}{4}$ corresponds to the ray $y=x$, $x\geqslant 0$. In this event,
we have

$$f(x,y) = \frac{2x^2}{x^2+x^2} = 1 \qquad (\text{if } x\neq 0) \qquad .$$

Therefore, on $y=x$, $x\geqslant 0$, we have

$$\lim_{\substack{(x,y)\to(0,0)}} f(x,y) = 1$$

which checks with (3).

c. Along the ray $\theta=0$, (2) yields

$$\lim_{\substack{(x,y)\to(0,0)}} f(x,y) = \sin 2(0) = 0$$

As a check this is the ray $y=0$, $x\geqslant 0$. In this event

3.1.7 continued

$$\frac{2xy}{x^2+y^2} = \frac{0}{x^2}$$

therefore

$$\lim_{(x,y)\to(0,0)} \left(\frac{2xy}{x^2+y^2}\right) = 0 \qquad .$$

When $\theta = \frac{\pi}{2}$, $2\theta = \pi$ and $\sin 2\theta = 0$. Therefore, on the ray $x=0$, $y \geqslant 0$;

$$\frac{2xy}{x^2+y^2} = \frac{0}{y^2}$$

therefore,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = 0 \quad \text{in this case also.}$$

Comparing (b) and (c) we see that while $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ if $(x,y)\to(0,0)$ either along the x-axis or the y-axis,

$\lim_{(x,y)\to(0,0)} f(x,y) = 1$ along the line of approach $\theta = \frac{\pi}{4}$. This shows that (1) $\lim_{(x,y)\to(0,0)} f(x,y)$ <u>does not</u> exist since its value depends on the path by which $(x,y)\to(0,0)$, and as an aside, (2) merely knowing that $f(x,y)$ approaches the same limit as (x,y) approaches $(0,0)$ along either the x- or the y-axis is not enough to say that this common value is $\lim_{(x,y)\to(0,0)} f(x,y)$.

3.1.8(L)

In a way, this exercise is a corollary to the previous one, but it is of enough significance in terms of what comes next in our course that we wanted to present the idea here.

3.1.8(L) continued

In our later work, expecially with 2-tuples, we shall select two
paths (four, counting sense) by which (x,y) will approach (a,b),
and we shall try to do almost everything else in terms of what
happens along these two paths. The paths will be (1) we hold
y constant and let x approach a, then we hold x constant and let
y approach b; (2) we hold x constant and let y approach b, then
we hold y constant and let x approach a. These two paths are
indicated below.

(1)   (2)
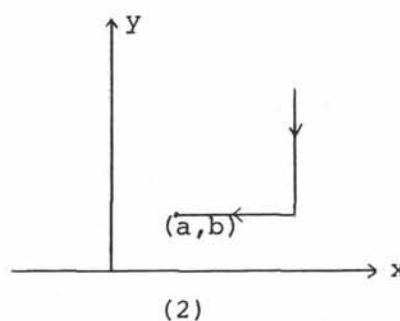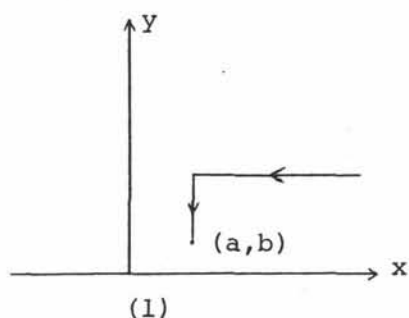
Now, as we said in Exercises 3.1.6 and 3.1.7, the limit of f(x,y)
as (x,y) approaches (a,b) may be different along these two paths.

The point is that it should not be surprising that the limit
depends on the path as well as the point of evaluation of the
limit; yet, as we get further into the course, we shall become
more and more preoccupied with the two special paths discussed
above. What will happen is that we shall have, in general,
sufficiently smooth functions so that what happens along these
two special paths will govern what happens along any other path,
but barring this "sufficient smoothness," there is no guarantee
that what happens along these two special paths is enough to tell
us what happens along every path.

This exercise is meant as a forewarner; it is designed to make
sure that we understand that when we say "independent of the
path" we mean more than just either the horizontal or vertical
directions.

3.1.8(L) continued

Based on the fact that f as defined in the previous exercise was
not defined at (0,0), coupled with the fact that the limit of f
as (x,y) approached (0,0) along the rays $\theta=0$ and $\theta = \frac{\pi}{2}$ was 0,
we are motivated to define a new function g by:

$$g(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \tag{1}$$

In other words, the g of this exercise and the f of the previous
exercise are _identical_ provided that $(x,y) \neq (0,0)$, but at (0,0)
f is not defined while g _is_ defined, and, in particular, g(0,0)=0.

a. We already know from the previous exercise that g is _not_ con-
    tinuous at (0,0), since among other things we showed that if
    (x,y) approached (0,0) along the ray $\theta = \frac{\pi}{4}$, then

$$\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2} = 1 \tag{2}$$

Notice that the meaning of "limit" in (2) tells us that
$(x,y) \neq (0,0)$, and under this condition, (1) tells us that

$$\frac{2xy}{x^2+y^2} = g(x,y).$$

Hence, (2) may be replaced by

$$\lim_{(x,y) \to (0,0)} g(x,y) = 1 \neq g(0,0) \text{ , since } g(0,0) = 0. \tag{3}$$

3.1.8(L) continued

Thus, at least along the path defined by $\theta = \frac{\pi}{4}$

$$\lim_{(x,y)\to(0,0)} g(x,y) \neq g(0,0) \qquad . \tag{4}$$

Equation (4) tells us that g is not continuous at (0,0) since if it were

$$\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0)$$

along <u>every</u> path connecting to (0,0).

b. What we want to show here is that

$$\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0)$$

if the approach is along either axis.

If we approach (0,0) along the x-axis, y=0 for the entire path. Thus we obtain, along this path,

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \\
&= \lim_{\substack{x\to0 \\ y=0}} \frac{2xy}{x^2+y^2} \\
&= \lim_{x\to0}^{*} \frac{0}{x^2} \\
&= 0 \\
&= g(0,0) \tag{5}
\end{aligned}
$$

---

*Recall that x→0 includes both x→0⁺ and x→0⁻. In terms of polar coordinates, we must check that the limits exist and are equal for the <u>two</u> rays θ=0 and θ=π. In this case, since g(x,y) is sin 2θ for $\overline{(x,y)} \neq$ (0,0), but θ=0 and θ=π make g(x,y) = 0 as (5) indicates.

3.1.8(L) continued

Similarly, approaching (0,0) along the y-axis yields

$$\lim_{(x,y) \to (0,0)} = \lim_{\substack{x=0 \\ y \to 0}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{y \to 0} \frac{0}{y^2}$$

$$= 0 = g(0,0)$$

This example shows once and for all that if

$$\lim_{(x,y) \to (0,0)} g(x,y) = g(0,0) \tag{6}$$

as $(x,y) \to (0,0)$ along either axis, we cannot guarantee that g is
continuous at (0,0). (Trivially, of course, if we know that g is
continuous at (0,0) then in particular (6) must be true. That is,
once we know that $\lim_{(x,y) \to (0,0)} g(x,y) = g(0,0)$ for <u>every</u> path,
then we know that it's true for a particular path; if, however,
we know what the limit is for some particular path, we cannot
necessarily conclude what it is for every path.)

3.1.9(L)

Our main aim here is to show how much more complicated functions
of several variables are than functions of a single variable.
We have deliberately chosen the case n=2 here so that (1) we can
again capitalize on any geometric interpretations and (2) we can
show that the complications do not require that we have more than
the "usual" 3-dimensional space.

The proof is not really too difficult once we get the hang of
things. (If you have difficulty visualizing our approach, perhaps
if, as we proceed, you refer to Figures 1 and 2, things will seem
clearer.) To begin with, if f happens to be a constant function

3.1.9(L) continued

(i.e., $f(x_1,y_1) = f(x_2,y_2)$ for every pair of points (x_1,y_1)
and (x_2,y_2) in the domain of f), the assertion of the theorem
follows trivially, for in this case there is a number c such that
$f(x,y) = c$ for all (x,y) in the domain of f.

Thus, the interesting case is when f is not a constant function.
Assuming, therefore, that f is not a constant function, there are
at least two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ for which $f(P) \neq f(Q)$.
Now pick any number, subject only to the condition that it lie
between $f(P)$ and $f(Q)$. That there are such numbers is guaranteed
by the fact that $f(P) \neq f(Q)$. Let us designate one such number
by m. Let C denote any curve that connects P and Q. Since f
is continuous, there must be some point on C, say X, for which
$f(X) = m$, since a continuous function takes on all its intermediate
values.

Of course, while you are likely to accept this last statement as
being true (or at least, as being reasonable), the fact remains
that we proved it only in the case of a single real variable.
This proof can be extended to the case of functions of several
real variables, but it is beyond our present purposes to become
so computationally involved at this time. While the interested
reader should feel free to prove this result on his own if he so
desires, we will use a geometrical interpretation. Recall that
in this case the domain of f is the entire xy-place. Now suppose
that f is continuous. This means, pictorially, that the surface
$z = f(x,y)$ is unbroken. Again, to add concreteness to our dis-
cussion, let us replace such literal points as P and Q by the more
specific points, say, $P(1,2)$ and $Q(3,4)$. Suppose, without loss of
generality, that $f(1,2) = 6$ while $f(3,4) = 9$. Now let C be any
continuous curve that joins P to Q. The path generated by a line
parallel to the z-axis running along C gives us a cylinder. (Recall
that mathematically a cylinder is obtained by tracing along any
curve in the plane with a line perpendicular to the curve), and
this cylinder intersects the surface $z = f(x,y)$ to form an un-
broken (space) curve. All we are now saying is that since

3.1.9(L) continued

this space curve is unbroken, and it passes through the heights
6 and 9, it must pass through every height between 6 and 9.
Stated in terms of planes, we are saying that our space curve
originates in the plane $z = 6$ and terminates in the plane
$z = 9$. Since the curve is unbroken, it must intersect every
plane $z = k$ where $6 < k < 9$.

Thus, still using our particular example, since $6 \leqslant 8 \leqslant 9$,
there is a point P_1 on the continuous curve C_1 which connects
$(1,2)$ and $(3,4)$ for which $f(P_1) = 8$.

Notice that C_1 was <u>any</u> curve that connected $(1,2)$ and $(3,4)$.
The point is that there are <u>infinitely</u> many such continuous
curves <u>which have no points other than $(1,2)$ and $(3,4)$ in common.</u>
For each of the curves C_n, there is a point P_n for which $f(P_n)=8$.
Since the curves intersect only at $(1,2)$ and $(3,4)$ the point
P_i on C_i cannot be the same as the point P_j on C_j (no P_n can
be either $(1,2)$ or $(3,4)$ because $f(P_n) = 8$ while $f(1,2) = 6$
and $f(3,4) = 9$). Thus, we have infinitely many different points
P_1, P_2,...,P_n,.... for which $f(P_1) = f(P_2) =...=f(P_n) = ... = 8$.
An attempt to summarize these results pictorially is made below:

3.1.9(L) continued

1. C_n is a continuous curve in the xy-plane connecting P(1,2) and Q(3,4).

2. The (right) cylinder along C_n intersects the surface z=f(x,y) along the space curve P'Q'.

3. P' is in the plane z=6 since its coordinates are (1,2,6) Q' is in z=9 since its coordinates are (3,4,9).

4. The plane z=8 intersects P'Q' at, at least, one point Q_n since P'Q' is a continuous curve.
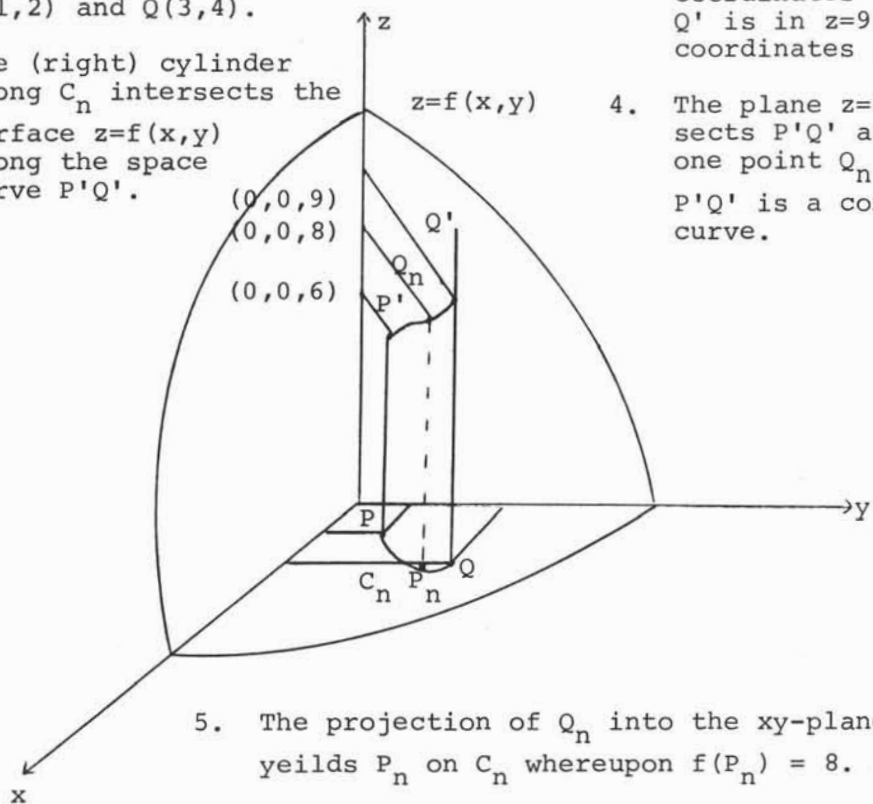
5. The projection of Q_n into the xy-plane yeilds P_n on C_n whereupon $f(P_n) = 8$.

(Figure 1)

The important thing in Figure 1 is that C_n was any continuous curve that joins P to Q. There are infinitely many such curves with no points of intersection other than P and Q. In Figure 2 four such curves C_1, C_2, C_3, C_4 are drawn. By the construction in Figure 1, we may obtain P_1 on C_1, P_2 on C_2, P_3 on C_3, P_4 on C_4 such that $f(P_n) = 8$ for every n = 1,2,3,4.

3.1.9(L) continued
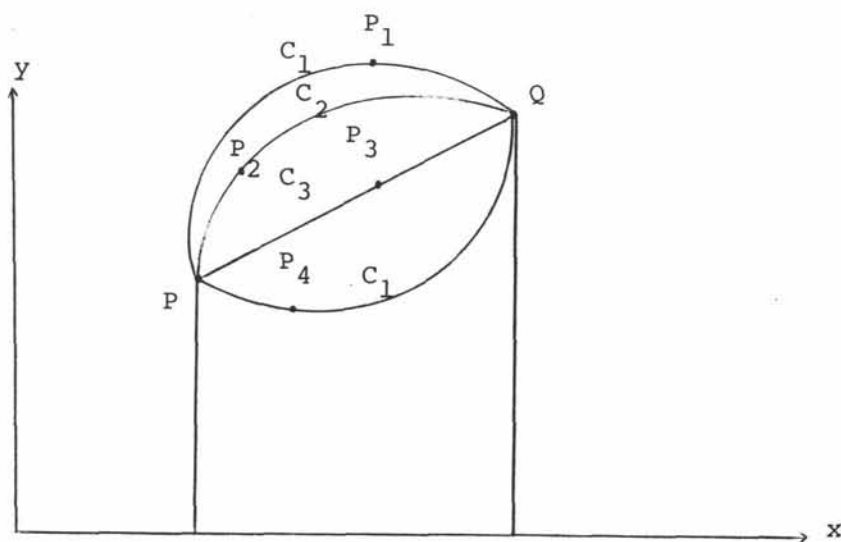
(Figure 2)

The point we now wish to make actually begins with the solution of
this exercise. We have now shown that if $f:E^2 \to E$ is any continuous
function, then it is impossible for f to be 1-1. In fact, in the
particular illustration we used, there were infinitely many points
in the domain of f at which f was 8.

What this exercise shows is that if the mapping from n-dimensional
space (with n>1) into 1-dimensional space is continuous, then f
cannot be 1-1. Restated more formally:

If n>1 and $f:E^n \to E$ is continuous then f is not 1-1, and, in fact,
there can be infinitely many elements in E^n which have the same
image in E.

(Notice that our argument holds if the domain of f is any subset
of E^n. That is, all that our argument required was that
$f(P) \neq f(Q)$ no matter how close to each other P and Q might be.)

Thus, we should begin to feel that while much of the theory for
functions of n-tuples is a generalization of the results for the
1-dimensional case, there are enough new "wrinkles" when n>1 to

3.1.9(L) continued

remind us to be on guard. Among other things, there are infinitely
many continuous functions $f:E^1 \to E$ which are 1-1 (namely those
whose graphs are either monotonically rising or monotonically
falling).

3.1.10

By now, we hope that it is virtually self-evident to you that n=4
was chosen simply for concreteness and that our approach extends
verbatim to any value of n.

To prove that "=" is an equivalence relation we must show that

(1) $\underline{a} = \underline{a}$ for each $\underline{a} \varepsilon E^4$

(2) $\underline{a} = \underline{b} \to \underline{b} = \underline{a}$, $\underline{a}$ and $\underline{b}$ in E^4

(3) $\underline{a} = \underline{b}$ and $\underline{b} = \underline{c} \to \underline{a} = \underline{c}$

As for (1), we have $\underline{a} = (a_1, a_2, a_3, a_4)$ where a_1, a_2, a_3, a_4 are all
<u>real</u> numbers. By the properties of equality for real numbers, we
know that

$$a_1 = a_1, \ a_2 = a_2, \ a_3 = a_3, \text{ and } a_4 = a_4.$$

Hence, by definition of vector (n-tuple) equality, we have

$$(a_1, a_2, a_3, a_4) = (a_1, a_2, a_3, a_4)$$

or

$$\underline{a} = \underline{a}$$

which establishes the validity of (1).

(Notice how (1) was established for E^4 by virtue of the corresponding
property of equality for real numbers).

3.1.10 continued

As for (2), let $\underline{a} = (a_1,a_2,a_3,a_4)$ and $\underline{b} = (b_1,b_2,b_3,b_4)$. Then, by definition, $\underline{a} = \underline{b}$ means

$$a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3, \quad \text{and} \quad a_4 = b_4$$

But, for real numbers, equality is symmetric. Hence, we may conclude that

$$b_1 = a_1, \quad b_2 = a_2, \quad b_3 = a_3, \quad \text{and} \quad b_4 = a_4$$

whence, by definition,

$$(b_1,b_2,b_3,b_4) = (a_1,a_2,a_3,a_4)$$

or

$$\underline{b} = \underline{a}$$

which establishes the validity of (2).

Finally, to establish the validity of (3), we let

$$\underline{a} = (a_1,a_2,a_3,a_4), \quad \underline{b} = (b_1,b_2,b_3,b_4) \quad \text{and} \quad \underline{c} = (c_1,c_2,c_3,c_4).$$

Then, from $\underline{a} = \underline{b}$ we conclude that

$$a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3 \quad \text{and} \quad a_4 = b_4, \tag{1}$$

while from $\underline{b} = \underline{c}$ we conclude that

$$b_1 = c_1, \quad b_2 = c_2, \quad b_3 = c_3, \quad \text{and} \quad b_4 = c_4 \tag{2}$$

Since for real numbers, $a = b$ and $b = c \rightarrow a = c$, we may conclude from (1) and (2) that

$$a_1 = c_1, \quad a_2 = c_2, \quad a_3 = c_3 \quad \text{and} \quad a_4 = c_4 \quad .$$

3.1.10 continued

Hence

$$(a_1, a_2, a_3, a_4) = (c_1, c_2, c_3, c_4)$$

or

$$\underline{a} = \underline{c} \quad .$$

Notice how this proof augments our remark in the supplementary notes that the fact that = is an equivalence relation on E^4 is induced by the fact that vectors may be viewed as n-tuples, each of whose components is real, and equality is an equivalence relation on the real numbers.

3.1.11
_____

(Note: The technique used in this specific exercise is the one used whenever we decide to view vectors in terms of their components).

We have, by definition of the arithmetic of E^4, that

$$\underline{b} + \underline{c} = (b_1, b_2, b_3, b_4) + (c_1, c_2, c_3, c_4)$$

$$= (b_1 + c_1, \ b_2 + c_2, \ b_3 + c_3, \ b_4 + c_4)$$

Hence, by our definition of dot product

$$\underline{a} \cdot (\underline{b} + \underline{c}) = (a_1, a_2, a_3, a_4) \cdot (b_1 + c_1, b_2 + c_2, b_3 + c_3, b_4 + c_4)$$

$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) + a_4(b_4 + c_4) \qquad (1)$$

Since the a_k's, b_k's and c_k's (k=1,2,3,4) are numbers, we know that (1) may be rewritten as

$$a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3 + a_4 b_4 + a_4 c_4 \qquad ,$$

3.1.11 continued

which, in turn, is

$$(a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4) + (a_1 c_1 + a_2 c_2 + a_3 c_3 + a_4 c_4) =$$

$$\underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$

which proves the required result.

Again, notice how we effectively reduce the study of E^4 to four separate studies of E^1 (i.e., the four components of any vector [n-tuple] in E^4). That is, the vector result is induced by the corresponding scalar result.

MIT OpenCourseWare
http://ocw.mit.edu

Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.