

Unit 3: Inverse Matrices

1. Lecture 4.030 (This lecture will also apply to Unit 4.)

Inverting a Matrix

Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

$y_1 = x_1 + x_2 + x_3$
 $y_2 = 2x_1 + 3x_2 + 4x_3$
 $y_3 = 3x_1 + 4x_2 + 6x_3$

Solve for x 's in terms of y 's

$A \begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 \\ 3 & 4 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}$

Claim: $A^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

since A^{-1} codes

$x_1 = 2y_1 - 2y_2 + y_3$
 $x_2 = 3y_2 - 2y_3$
 $x_3 = -y_1 - y_2 + y_3$

a.

Matrix Algebra Interpretation

$y_1 = x_1 + x_2 + x_3$
 $y_2 = 2x_1 + 3x_2 + 4x_3$
 $y_3 = 3x_1 + 4x_2 + 6x_3$

is the single matrix equation $Y = AX$

$\therefore A^{-1}Y = A^{-1}(AX) = X; X = AY$

Function Interpretation

System (1) may be viewed as $f: E^3 \rightarrow E^3$ where $f(x_1, x_2, x_3) = (y_1, y_2, y_3)$ [or $f(x) = y$]

e.g. $f(1, 1, 1) = (3, 9, 13)$
 $f(-1, 2) = (2, 9, 13)$

f and A are "identifiable" since $y = f(x)$ and $Y = AX$ convey equivalent information.

\therefore Existence of A^{-1} is equivalent to existence of f^{-1}

e.g. $(-1, 2) \xrightarrow{f^{-1}} (2, 9, 13)$

\therefore Since f^{-1} need not exist for a given f , A^{-1} need not exist for a given matrix, A .

In other words,

b.

Not all matrices are invertible

Example

Try to find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

A codes

$y_1 = x_1 + x_2 + x_3$
 $y_2 = 2x_1 + 3x_2 + 4x_3$
 $y_3 = 3x_1 + 4x_2 + 5x_3$

$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$

$\therefore x_1 - x_2 = 3y_1 - y_2$
 $x_2 + 2x_3 = -2y_2 + y_3$
 $-y_3 - y_2 = 0$

\therefore (i) Unless $y_3 = y_2$, we cannot express x 's in terms of y 's

(ii) If $y_3 = y_2$, then, for example, x_3 is independent of y_1, y_2 and y_3 .

c.

Lecture 4.030 continued

Function Interpretation
 If $f: E^3 \rightarrow E^3$ is defined by system (3):

(1) For $y = (1, 1, 3)$ there is no x such that $f(x) = y$ since $3 \neq 1+1$
 $\therefore f$ is not onto

(2) If $y = (1, 1, 2)$ system (4) applies
 $\therefore x_1 - x_3 = 3y_1 - y_2 = 2$
 $x_2 + 5x_3 = -2y_2 - y_3 = -1$

$\therefore x_1 = x_3 + 2$
 $x_2 = -2x_3 - 1$
 $\therefore f(x_3 + 2, -2x_3 - 1, x_3) = (1, 1, 2)$
 $\therefore f$ is not 1-1

Pictorially

Geometric Note
 $f: E^3 \rightarrow E^3$ may be viewed as:

f maps 3-space into the plane $y_3 = y_1 + y_2$
 $f(x)$ is the point $(1, 1, 2)$

d.

Summary

Part I
 Given the n by n matrix A , form the n by $2n$ matrix $[A: I_n]$
 i.e. $\begin{bmatrix} a_{11} & \dots & a_{1n} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} & \dots & 1 \end{bmatrix}$
 Row-reduce this matrix. Then either (i) the left-half contains at least one row 0's, whence A^{-1} doesn't exist; or (ii) the left-half reduces to I_n , whence the right-half is A^{-1} .

Part II
 If A^{-1} exists, then in terms of matrix algebra, we can solve (invert) the equation $Y = AX$ to conclude $X = A^{-1}Y$

In terms of mappings, let f be defined by $f(x_1, \dots, x_n) = (y_1, \dots, y_n)$ where $\begin{cases} y_1 = a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ y_n = a_{n1}x_1 + \dots + a_{nn}x_n \end{cases}$
 Then f^{-1} exists $\iff A^{-1}$ exists, where $A = [a_{ij}]$
 If A^{-1} doesn't exist then f is neither 1-1 nor onto

e.

Study Guide
Block 4: Matrix Algebra
Unit 3: Inverse Matrices

2. Read Supplementary Notes, Chapter 6, Section E.

3. (Optional) Read Thomas, Section 13.3.

4. Exercises:

4.3.1

Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, C = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}, 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Solve each of the following matrix equations.

a. $\frac{1}{3}X - AB = C.$

b. $AX - BC = 0.$

4.3.2

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and let } X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

Assuming that $ad - bc \neq 0$, solve $AX = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

4.3.3

Compute A^{-1} if $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, imposing any necessary conditions on a or d to insure the existence of A^{-1} . Generalize this result to find A^{-1} if

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

(i.e., $A = [a_{ij}]$ where $a_{ij} = 0$ when $i \neq j$).

4.3.4(L)

- a. Let A and B denote any non-singular 2 by 2 matrices (i.e., A^{-1} and B^{-1} exist). Show that AB is also non-singular and in particular $(AB)^{-1} = B^{-1}A^{-1}$. (Note the order of the factors.)
- b. Check (a) when $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$.

4.3.5

Let A denote the matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 3 & 9 & 7 \end{pmatrix}$$

- a. Use the augmented-matrix technique described in the notes to find A^{-1} .
- b. If

$$y_1 = x_1 + 3x_2 + 5x_3$$

$$y_2 = 2x_1 + 7x_2 + 9x_3$$

$$y_3 = 3x_1 + 9x_2 + 7x_3$$

express x_1 , x_2 , x_3 as linear combinations of y_1 , y_2 , and y_3 .

- c. Solve the system of equations

$$x_1 + 3x_2 + 5x_3 = 8$$

$$2x_1 + 7x_2 + 9x_3 = 16$$

$$3x_1 + 9x_2 + 7x_3 = 32$$

4.3.6

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 3 & 9 & 7 \end{bmatrix}, B = \begin{bmatrix} 0 & 4 & 8 \\ 8 & 2 & 16 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 8 \\ 4 & 2 \\ 8 & 16 \end{bmatrix}$$

- a. Determine X if $AX = C$.
b. Determine Y if $YA = B$.

c. Repeat parts (a) and (b) with $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 8 \end{bmatrix}$.

d. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Show that there exist infinitely many matrices X such that $AX = I_2$.

- e. With A , I_2 , and I_3 as in part (d), show that there is no matrix X such that $XA = I_3$.

4.3.7

Find the flaw in the following argument. "Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Replace the first row of A by the sum of the first and the second rows; and replace the second row of A by the sum of the second row and the first row. Call the resulting matrix B . Therefore, $B = \begin{pmatrix} 4 & 6 \\ 7 & 10 \end{pmatrix}$. Since B was obtained from A by the 'permitted' operations described in the notes, it follows that $A \sim B$."

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Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

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