AN INTRODUCTION TO VECTOR ARITHMETIC

A

Introduction

In the formula $V = \pi r^2 h$, we observe that to find V we must know the values of both r and h. In terms of a function machine, we may think of the input as being the ordered pair of numbers r and h, written, say, as (r,h). The output is obtained by multiplying the square of the first member by the second member and then multiplying the resulting product by π. We say ordered pair since it certainly makes a difference in general whether it is r or h that is being squared. For example, the input (3,4) yields $\pi 3^2 4 = 36\pi$ as an output, whereas (4,3) yields $\pi 4^2 3 = 48\pi$ as an output.

While we shall deal with this idea in greater detail as our course unfolds, let us say for now that an ordered pair of numbers will be called a 2-tuple and that, quite in general, an n-tuple will refer to an ordered sequence of n numbers.

In this context, the study of functions of several real variables involves a study of n-tuples. For example, given the equation

$$y = x_1^2 + 2x_2^3 + 4x_3 + 5x_4$$

we see that to determine y it is required what we know the values of x_1, x_2, x_3, and x_4. In this context, the y-machine, which yields y as an output, has the 4-tuple (x_1, x_2, x_3, x_4) as the input. As a specific illustration, if (4,2,1,6) is the input, the output is $4^2 + 2(2)^3 + 4(1) + 5(6) = 66$.

Thus, as we hope to make more clear as we go along, the study of n-tuple arithmetic is a rather important aspect of functions of several variables.

What is also interesting is that, in the case of 2-tuples and 3-tuples, there is a rather interesting structural connection with the usual 2- and 3-dimensional vectors of physics and engineering. While we wish to go beyond the traditional 2- and 3-dimensional case is favor of the general n-tuple case, the fact is that ordinary vectors are interesting and important in their own right, and, at the same time, by getting familiar with the tangible 2- and 3-dimensional cases, we build our experience so that we may better handle the more abstract n-tuple arithmetic when n is greater than 3.

With this in mind, we devote this chapter first to reviewing the traditional idea of vectors and second to showing how vector arithmetic is structurally related to numerical arithmetic as studied in the previous chapter.

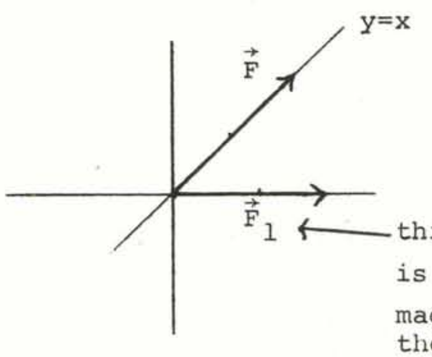
In the next chapter we shall extend our ideas to that of the calculus of vectors, after which we branch out to the more general calculus of n-dimensional space (the name usually given to the arithmetic of n-tuples).

At any rate, with the overview in mind, we now turn to a discussion of "traditional" vectors.

B

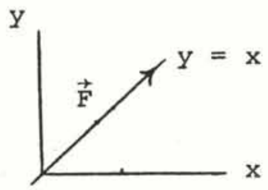
Vectors Revisited

Certain quantities depend on direction as well as magnitude. For example, if the distance from town A to town B is 400 miles and we leave A and travel at 50 miles per hour we will arrive at B in 8 hours provided we travel in the direction from A to B. A quantity which depends on direction as well as magnitude is called a vector (as opposed to a quantity which depends only on magnitude, which is called a scalar). A rather natural question is: how shall we represent a vector quantity pictorially? Recall that for scalar quantities we have already agreed to use the number line and identify numbers with lengths. That is, we think of the number 3 as a length of 3 units. We generalize this idea to represent vector quantities. Namely, we agree that rather than use only the length of a line (as we do with scalars) we will use the notion of directed length as well. That is, given the vector which we wish to represent, we draw a line in the direction of the vector, whereupon we then choose the length of the line to represent the magnitude of the vector. For example, if we wish to indicate a force of 2 pounds acting in the direction of the line $y = x$, we would represent it as the arrow $\vec{F}$.

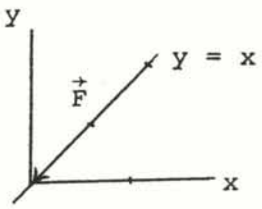


this also represents a force whose magnitude is 2 pounds, but, while $\vec{F}$ and $\vec{F}_1$ have equal magnitudes, they are different forces since they act in different directions.

Notice from our diagram that there is still one ingredient that must be defined if we wish to represent our vector without ambiguity. We must indicate the sense of the vector. That is, in our present example, do we mean



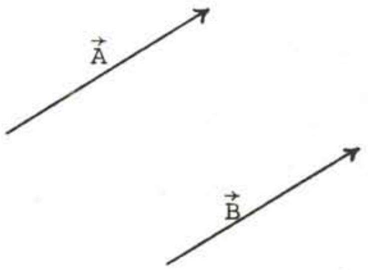
or do we mean



both vectors have magnitude equal to 2 and both act in the direction of $y = x$.

We only talk about sense once the direction is known. In other words, we do not ask whether two vectors have the same sense until we know whether they act in the same direction. To indicate the sense we use an "arrowhead." Thus, we may say that the geometric interpretation of a vector is as an arrow. It is important to note (in much the same vein as our discussion in Part 1 of Calculus of the difference between a function and its graph) that the arrow is the geometric representation of the vector not, the vector itself. For example, force and velocity are not arrows but merely representable as arrows. Of course, once we have made this distinction, we may use arrows and vectors more or less interchangeably in the same way we use graphs and functions.

In any event, we are now ready to begin our "game" of vector (or arrow) arithmetic. So far, all we have is a set of objects called vectors. We do not, as yet, have a structure. A mathematical structure is more than a set. It is a set together with various rules and operations. For a start, we would most likely prefer to define some equivalence relation so that we can determine when two vectors can be called equivalent (equal). Since the only important ingrediants of a vector are its magnitude, sense, and direction, we agree to call two vectors (arrows) "equal" if and only if they have the same magnitude, sense and direction. In particular, this definition of equality means that if we have two distinct parallel line segments of the same length and we orient them with the same sense then these two arrows are equal. Pictorially,

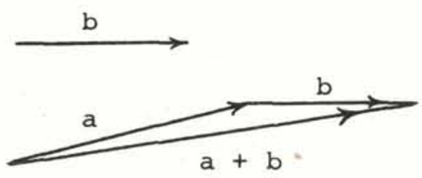


$\vec{A}$ and $\vec{B}$ are called equal and we write $\vec{A} = \vec{B}$. Notice that as lines they would be parallel but different.

If this usage of equality strikes you as being rather odd, notice that we have used this idea as early, for example, as when we make such remarks as $\frac{1}{2} = \frac{2}{4}$. Clearly, these two symbols do not look alike. What we mean is that if $\frac{a}{b}$ is defined to mean that number which when multiplied by b yields a, then $\frac{1}{2} = \frac{2}{4}$ since <u>the</u> number which must be multiplied by 2 to yield 1 is the same as <u>the</u> number which must be multiplied by 4 to yield 2. On the other hand, if we agree to use $\frac{a}{b}$ as an exponent wherein the numerator tells us what power to raise to and the denominator tells us the root to abstract, then $\frac{1}{2}$ and $\frac{2}{4}$ are no longer equal since $1^{1/2} = \{1, -1\}$, while $1^{2/4} = \{1, -1, i, -i\}$. In other words, equality is always with respect to a particularly defined relation. In any event, notice that this definition of the equality of two vectors is an equivalence relation. That is, (1) any vector is equal to itself, (2) if the first vector equals the second then the second equals the first, and (3) if the first vector equals the second and the second equals the third, then the first equals the third. Hence, if we go back to rules E-1 through E-5 in Section D of Chapter 1, we see that these rules remain "true" if we everywhere replace the word "number" by the word "vector." (For convenience in referring, we shall use a and b rather than $\vec{a}$ and $\vec{b}$ when referring to our 16 rules.)

The next problem in developing our structure is to impose a <u>binary operation</u> on the set of vectors. This can, of course, be done in many different ways. In other words, there are an endless number of ways in which we can make up rules whereby we can combine vectors to form vectors. Rather than proceed randomly (and here is a natural connection between pure and applied mathematics) we choose as our rule one that we believe to be true in the real world. In particular, the physical notion of the <u>resultant</u> vector motivates us to choose our binary operation. Stated in terms of arrows, we define the <u>sum</u> of the vectors a and b (written a + b)* as follows:

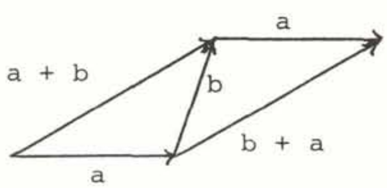
We shift b parallel to itself without changing its magnitude and sense, (notice that by our definition of equality the "new" vector which results from this shift is equal to b) until its "tail" coincides with the "head" of a. Then a + b is merely the arrow that starts at the tail of a and terminates at the head of b. Pictorially,



Leaving out the a and b notation we can say: to add two vectors, we place the tail of the second at the head of the first, and the sum is then the arrow which goes from the tail of the first to the head of the second.

Again, it is important to note that we were forced to pick this definition, but by making this choice we are sure that our definition has at least one real interpretation, namely the usual idea of a resultant.

At this stage of the development, we are at least sure that our concept of "sum" is a binary operation since it tells us how to determine an arrow from two given arrows. We now check to see what properties our "sum" has. Our first contention is that for any pair of vectors a and b, a + b = b + a. To check that this property holds, we need only compare a + b and b + a and observe what happens. To this end:

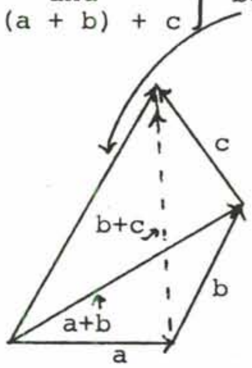


a + b and b + a are opposite sides of a parallelogram and have the same sense. Therefore, a + b = b + a.

*It is customary to refer to "sum" and to use the symbol + to denote a binary operation. We must be broadminded enough, mathematically speaking, not to think of "sum" as being restricted to the ordinary sum of two numbers. Actually, however, if we wished to be completely rigorous and unambiguous, we should have invented a new symbol, for example, a*b rather than a + b to denote the binary operation whereby we combine two vectors to form a vector.

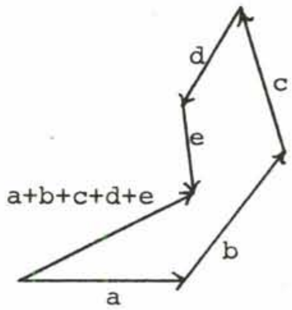
We now check to see whether for three vectors a, b, and c, $a + (b + c) = (a + b) + c$. We observe:

$a + (b + c)$ and $(a + b) + c$ } both name this vector



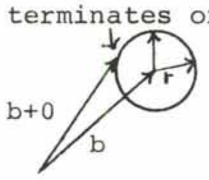
Notice, therefore, that a + b + c is unambiguous, and it is the vector that goes from the tail of <u>a</u> to the head of c when a, b, and c are properly aligned head to tail.

What this means is that we do not have to worry about "voice inflection" when we "add" vectors, or, in other words, if a, b, c, d, and e denote vectors, the expression $a + b + c + d + e$ <u>unambiguously</u> names a vector. Pictorially, this becomes the "polygon rule," a generalization of the parallelogram rule illustrated earlier.



Notice that, so far, if we continue to replace the word "number" by "vector," rules A-1, A-2, and A-3 still apply to our game of vectors.

Our next step is to see whether there is a vector which plays a role of additive identity. That is, given any vector b, is there a vector, denoted by 0, such that $b + 0 = b$. In terms of our arrow interpretation, it is clear that if the vector 0 were to have a non-zero length then b and $b + 0$ could not possibly have the same length. Again, pictorially,

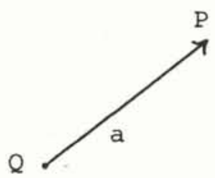


If 0 had magnitude $r \neq 0$ then $b + 0$ would have to terminate somewhere on the circle. And no point on the circle can be the center of the circle.

Thus, for the vector $\vec{0}$ to have the desired property, we must define it to be a vector whose magnitude is zero (i.e., the number 0), independently of any mention of direction or sense.

To be able to refer to the zero vector, (as we refer to the number zero), we must agree that we do not distinguish between two vectors of zero magnitude, even if they have different directions. This agrees with our geometric intuition, since a point has no direction. Thus, the definition of equality is waived for the zero vector, and we simply agree to call any two vectors of zero magnitude equal.

Finally, we want to investigate the notion of whether, given any vector a, we can find another vector b such that a + b = 0 (where 0 here denotes the vector 0*). Since the zero vector has no length and since we add vectors "head to tail" it follows that if a + b is to equal 0 then the tail of a and the head of b must coincide. This in turn means that we have the same magnitude and direction as a but the opposite sense. Pictorially,



If b originates at P it must terminate at Q if a + b = 0. Therefore, if we think of a as $\overrightarrow{QP}$ then b = $\overrightarrow{PQ}$.

In other words, if we wish vector addition to have the same structure as that of the numerical addition (and the choice is ours to make) we must define the inverse of a, i.e. (-a), to be the vector which has the same magnitude and direction as a but the opposite sense.

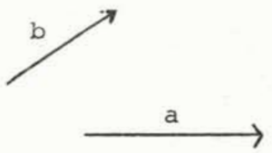
If we again agree, as in the case of numerical addition, that a + (-b) will be abbreviated by a - b, then to form the vector a - b we proceed as follows:

(1) a - b means a + (-b)

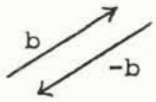
(2) To obtain (-b) from b simply reverse the sense of b.

(3) We now add a and (-b) in the "usual" way; the sum being a + (-b), or, therefore, a - b. Pictorially,

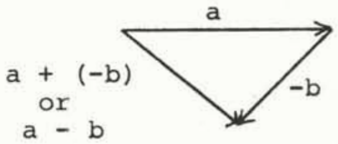
*Because there is a possibliity of confusing scalars and vectors in many cases, it is conventional to use different symbolisms for vectors and scalars. In some texts, one uses greek letters for vectors and "regular" letters for scalars; or one uses boldface type for vectors, or one writes arrows over the vector (such as $\vec{0}$). Later we will do this but for now we prefer to have our symbolism look as much like that in rules E-1 through A-5 as possible.



(i)



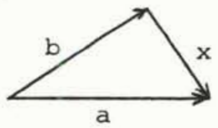
(ii)



(iii)

While (we hope!) that our explanation of subtraction is adequate and that it certainly shows the resemblance between the relationship, structurally, of vector subtraction and numerical subtraction, there is yet another way to view subtraction of vectors - a way that might be more easily remembered from a computational point of view.

Suppose we are given the vectors a and b and we now place them tail-to-tail. (If we desire a rationalization, if we add head-to-tail, why not subtract tail-to-tail?) Let us now look at the vector that extends from the head of b to the head of a, and just as in numerical arithmetic, let us label this "unknown" x. Thus,



In the above arrangement of vectors, only x and b are properly aligned (head-to-tail) for addition. That is, our diagram yields the "equation";

$$b + x = a. \qquad (1)$$

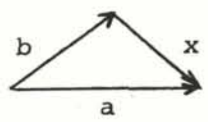
Had we not been told that (1) was a vector equation, and instead we treated equation (1) as if only numbers were involved, we would have obtained

$$x = a - b. \qquad (2)$$

In numerical arithmetic, the process of getting from (1) to (2) is called transposing, and its validity is established roughly along the lines of equals subtracted from equals are equal. It turns out that this property of transposing which got us from (1) to (2) is a property of our five rules for equality and our five rules for addition. [The proof is left to the interested reader, and involves writing $x + b = a$ as $(x + b) + (-b) = a + (-b)$.]

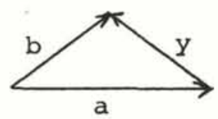
Since these same ten rules apply to vectors as well as numbers, our game-idea tells us that the process of getting from (1) to (2) is equally valid when we are dealing with vectors.

Summed up, then pictorially, to find the difference of two vectors, we place the two vectors tail-to-tail, and then draw a vector from the head of one to the head of the other. Notice that this can be done with two different senses, but the rule of transposing tells us which is which. That is:



$b + x = a$

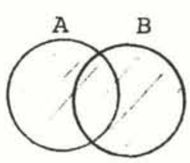
therefore $x = \underline{a - b}$



$$a + y = b$$
$$y = \underline{b - a}.$$

Summed up more formally, if a and b are any two vectors and we wish to find $a - b$, we place the two vectors tail-to-tail and $a - b$ is then the vector which goes from the head of b to the head of a.

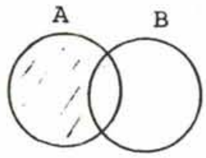
Again we must stress the importance of structure. The mere fact that we have a binary operation denoted by "+" and a statement that $a + b = c$, we must not jump to the conclusion that $a = c - b$. Transposing is a theorem in a structure that obeys some particular rules. As an example, let us again take "+" to mean union. Then if $a + b = c$, it is not necessarily true that $a = c - b$. Pictorially,



$C = A \cup B$
$(A + B)$

$C - B = \{x: x \in C, x \notin B\}$

therefore, C - B =



which is unequal to A unless the special case $A \cap B = \emptyset$ prevails.

Again, since the structure of sets with respect to union is different from the structure of numbers with respect to addition, this example is not any kind of contradiction. Rather it should caution us that while it is nice to transpose just because it seems "natural," we must not belittle the fact that it is only because of a particular structure that we can enjoy this privilege.

With these remarks behind us, we are now in a position to see how the structure of arithmetic really works. Notice that in the playing of our "game" it is the rules which tell us how the terms are related that are so important - not the terms themselves. For example, we have just seen that rules E-1 through A-5 which applied to "ordinary" arithmetic are also correct for vector arithmetic provided that everywhere the word "number" appears we substitute the word "vector." Consequently, any conclusion that follows inescapably from these ten rules in numerical arithmetic will be a valid conclusion in vector arithmetic as well.

One way to illustrate this idea is to take a proof which we have given in numerical arithmetic and reproduce it verbatim, but replace "number" by "vector," and observe that the given proof is still valid in the new situation. More specifically,

Theorem 1

If a, b, and c are vectors such that $a + b = a + c$ then $b = c$.

Proof

Statement	Reason
(1) There exists a vector $-a$	(1) A-5
(2) $-a + (a + b) = -a + (a + c)$	(2) E-4 (replacing $a + b$ by $a + c$)
(3) $-a + (a + b) = (-a + a) + b$ $-a + (a + c) = (-a + a) + c$	(3) A-3
(4) $(-a + a) + b = (-a + a) + c$	(4) Substituting (E-4) (3) into (2)
(5) $-a + a = a + (-a)$	(5) A-2
(6) $a + (-a) = 0$	(6) A-5

Proof cont'd

Statement	Reason
(7) $-a + a = 0$	(7) Substituting (E-4) (5) into (6)
(8) $0 + b = 0 + c$	(8) Substituting (E-4) (7) into (4)
(9) $\left.\begin{array}{l} b + 0 = 0 + b \\ c + 0 = 0 + c \end{array}\right\}$	(9) A-2
(10) $\left.\begin{array}{l} b + 0 = b \\ c + 0 = c \end{array}\right\}$	(10) A-4
(11) $0 + b = b$ $0 + c = c$	(11) Substituting (E-4) (9) into (10)
(12) $b = c$	(12) Substituting (E-4) (11) into (8)

q.e.d.

Notice that we copied that statement-reason proof word for word from the corresponding theorem in numerical arithmetic. However, if we were to show this proof to a person without telling him how we obtained it, the proof stands validly on its own without reference to the proof from which we copied it.

This idea can be quite readily generalized as follows. Let S denote a set and suppose "=" denotes any equivalence relation defined on S while "+" denotes any binary operation on S which obeys A-2 through A-5. (We omit A-1 since A-1 is automatically obeyed by virtue of the fact that "+" denotes a binary operation. That is, we defined a binary operation to be equivalent to the Rule of Closure.) Then, in this particular structure, if a, b, and c are any elements in S and if $a + b = a + c$, then $b = c$.* This result is an inescapable conclusion based on the assumptions encompassed by the ten rules E-1 through A-5. (In fact, if we want to be even more precise, we can argue that the result follows from a subset of these ten rules, since not all ten rules were used in the proof.)

*We cannot emphasize enough the idea that the structure must be the same. That is, our cancellation law depends on our ten rules. As a counter-example, suppose we think of "+" as denoting the union of sets. Then, if a, b, and c are sets, we cannot conclude that if $a + b = a + c$ then $b = c$. Indeed, in our exercises in Part 1 of our course we showed by example that this need not be true. (As a quick review, suppose b and c are unequal subsets of a. Then $a + b = a + c$ since both equal a, but $b \neq c$.) This does not contradict what we are saying above. Rather, structurally, the union of sets is not the same as the addition of numbers.

C

Scalar Multiplication

Up to now we have been emphasizing the resemblance between vector and scalar arithmetic. The fact that these two different structures resemble one another in part is no reason to suppose that they share all properties in common, and one rather elementary but interesting difference involves the process known as scalar multiplication whereby one multiplies a number (scalar) by a vector.

The basic definition is pretty much straightforward. If we multiply a vector by a number c the result is a vector whose direction is that of the original vector and whose sense is the same if c is positive but opposite if c is negative. The magnitude of the new vector is $|c|$ times the magnitude of the original vector. By way of illustration, $-2v$ is a vector whose magnitude is twice that of v and which has the same direction but opposite sense of v.

Structurally, the arithmetic properties of scalar multiplication that are of the most interest to us are:

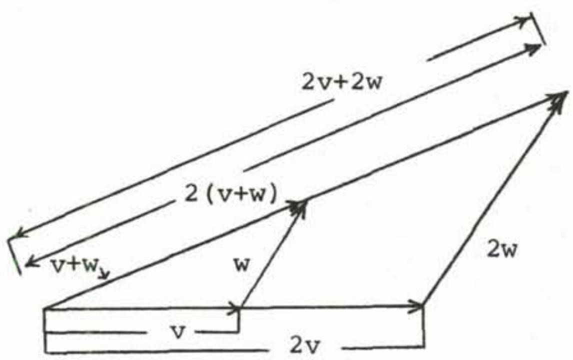
SM-1 If r and s are numbers and v is a vector then $r(sv) = (rs)v$

SM-2 If r and s are numbers and v is a vector then $(r + s)v = rv + sv$.

SM-3 If r is a number and v and w are vectors then $r(v + w) = rv + rw$.

SM-4 If r is any vector then $1r = r$.

In terms of our game of vectors (1), (2), (3) and (4) are the rules by which we play the game of scalar multiplication. That these rules are realistic follows from the fact that our model made of "arrows" obeys them. For example (3) is the geometric equivalent of similar triangles. By way of illustration we demonstrate geometrically that $2(v + w) = 2v + 2w$



Other details are left to the textbook and the exercises. The major point to observe is that our discussion of vectors, at least so far, is completely independent of any coordinate system. We should point out that it is often desirable to study vectors in Cartesian coordinates since in this system there is an interesting and rather simple form that the arithmetic of vectors takes on. This is explained very well in the text and we exploit these properties in our exercises. But it is important to understand that the concept of vectors transcends any coordinate system.