GILBERT STRANG: OK. This video is about the third of the great trio of partial differential equations. Laplace's equation was number one. That's called an elliptic equation. The heat equation was number two. That's called a parabolic equation. Now we reach the wave equation. That's number three, and it's called a hyperbolic equation. So somehow the three equations remind us of ellipses, parabolas, and hyperbolas.

They have different types of solutions. Laplace's equation, you solve it inside a circle or inside some closed region. The heat equation and the wave equation, time enters, and you're going forward in time. The heat equation is first order in time, du dt. And the wave equation, the full-scale wave equation, is second order in time. That stands for the second derivative, d second u dt squared. And it matches the second derivative in space with a velocity coefficient c squared.

I'm in one-dimensional space. If I were in three dimensions, where we really have sound waves and light waves and all the most important things in life, then there would be a uxx and a uyy and a uzz, second derivatives in all the space directions. But that's good enough to do 1D.

So what are the differences, first of all, between the heat equation and wave equation? So I'll say heat versus wave equations. What are the sort of biggest differences? The heat, the signal travels infinitely fast. Under the wave equation, the signal travels with finite velocity, and that velocity is that number c with speed c.

So fortunately, for sound waves, the wave comes to us, or when I'm speaking to you, my voice is traveling out to the microphone at the speed of sound, the speed c. And actually, another good thing is after it gets there, it goes on, and it goes by the point and doesn't just stay there messing up the future sounds. It travels. It dies out.

Whereas-- well, the heat. So let me try to give an example. Suppose the initial condition is a delta function. So if u equals a delta function, a point source, which is quite normal. A point source of heat, something really hot, or a point source of sound, my voice. So it's that at t equals 0. Then for the heat equation, there's a famous solution to the heat equation.

Remember, the heat equation is du dt equal uxx, first derivative in time. And the solution that starts from the delta function? Oh, do I know what it is? I think there's a 1 over-- there's a

square root of 4 pi t. There's an e to the minus x squared over 4t. I think perhaps that's it. Perhaps that's it.

So what do I see from that? I see big damping out. I see immediate travel. As soon as time is just a little beyond 0, then for every x we get an answer, but it's an extremely, extremely small answer. e to the minus x squared is tailing off to 0 incredibly fast. So that's a very small answer. Very little of the heat immediately gets very, very quickly across the ocean. But in theory, a little bit does.

Whereas for the wave equation, it takes time to cross the ocean. We have a tsunami. We have a wave. It gets there. It reaches the other side. And actually, at the-- it's very important. What is the speed of that wave to tell people about a tsunami that's coming, and you can actually do it, which you couldn't do for the heat equation.

So for the wave equation, what comes out of a delta function in 1D? Well, a wave goes to the right, and a wave goes to the left. That's what happens. And those waves are 1/2 of a delta function each way. So the solution is 1/2 of a delta function that's traveling. I see that-- let me write down the other half that's traveling the other way-- delta at x plus ct.

That's a cool solution. So that means the sound in 1D, the sound, half of it takes off in one direction and half in the other direction. And what happens in each direction is a spike of sound. You don't hear anything, then at a particular time, depending on your position x, there's a particular time when you get 0 in there, and you hear the signal. And then as soon as time goes past that, it's past you. So you get a big shock, and it comes to you with speed c.

If you look at that expression x minus ct, it's telling you that the speed of the wave in dx dt for the wave is c. OK. So that's a contrast for a very particular initial condition-- a big wall of water, a big noise, a big bang. OK.

I want to solve the wave equation, study it further for other initial conditions. And of course, initial condition's plural because the wave equation is second order in time. So I'm given u at t equals 0 and all x and du dt. I'm given an initial distribution of the wall of water, shall we say, and its velocity, the normal thing.

When I have a second-order equation, I'm given an initial condition and an initial velocity. And of course, because it's a partial differential equation, I'm given those for every x. So we have functions instead of just two numbers, and that's where a Fourier series can come in.

So we can solve that by a Fourier series if we're on a finite-- like a violin string. You've plucked a violin string that starts waves going back and forth in the string. They solve the wave equation. You hear music, great music if it's a good sound. Or we could solve it on an infinite line with no boundary, as in an essentially infinite ocean or waves in space. Light waves in space are solving the wave equation with no boundary, as far as we know.

OK. So which shall I do? I'll write the solution down in free space, and then I'll write one down for a violin string. So in space-- well, it's a one-dimensional space here. So this would be minus infinity less than x less than infinity.

Then what does the solution to the wave equation look like? It looks like some function of plus some function of x plus ct. Well, that's exactly the form that we had here when they were delta functions. Here, in general, they don't have to be delta functions. I chose that function f and that function g so that at t equals 0, I'm good. So at t equals 0-- so I set t equals 0. At t equals 0, u of 0 and x would be f of x plus g of x. Good.

But that's only one condition, and I have f and g to find. So I would also use du dt at the start. What's the time derivative? It's minus c f prime at x. And the time derivative of this will be a plus c g prime at x. OK. No big deal. I'm given two functions. I've got two functions to find in the answer, and I've got two equations. I could solve those for f and g, and it gives a formula called d'Alembert's formula, named after the person who put this together.

I'd rather go on to the violin, to a finite string. OK. So now I have a finite string, and I'm holding u equals 0 at the ends. OK. And my solution will still be functions that depend on x minus ct waves. My solution will still be waves.

The solution that I'm going to write down comes from a very, very important method called separation of variables. I want to separate x from t. I need to give you a full video about separation of variables. That's the best tool we have to get solutions to lots of equations.

So let me just jump to the form of the solution here. I'm imagining the violin string is at rest. So it starts with-- I'm imagining, let's say in our particular problem, this is 0, the initial velocity. This is plucked. The starting position is-- with your finger you've moved it off 0. OK.

What do the solutions look like, u of t and x? OK. They will be a sum of neat, special, convenient, separated solutions, with separated meaning t separated from x. And I think we would have cosine of nct. I'm going to have a sum. Oh, we'll have a coefficient, of course, a

sub-- well, or b, or c, or even d. How about d for a new letter-- d sub n. And I think-- well, let me finish here. I'm separating that from I think it would probably be the sine of nx. OK.

If we just look at that for a little bit, that's the point of this video. So this is x equals 0. This is, say, x equal pi. That's for convenience. Suppose our violin string has length pi, then, I think, that's what the solution looks like. It starts from at t equals 0. At t equals 0, the cosine is 1. So at t equals 0, this is an initial condition that we have to match. This'll tell us the d's will be the sum of dn sine nx. That tells us the d's. And then we have our answer.

So the initial condition-- remember, it's starting at rest. So the initial velocity is 0, and that's why I don't have any sine t's because I'm starting with initial velocity 0. I only have cosines in the t direction. But I only have sines in the x direction because the violin string is being held down at the two ends, and it's the sine function that matches that perfectly.

So this is separation of variables, t separated from x. And I really have to do a proper explanation of that highly important method of getting solutions like this. Thank you.