GILBERT STRANG: OK. I'm concentrating now on the key question of stability. Do the solutions approach 0 in the case of linear equations? Do they approach some constant, some steady state in the case of non-linear equations?

So today is the beginning of non-linear. I'll start with one equation. dy dt is some function of y, not a linear function probably. And first question, what is a steady state or critical point? Easy question. I'm looking at special points capital Y, where the right-hand side is 0, special points where the function is 0. And I'll call those critical points or steady states.

What's the point? At a critical point, here is the solution. It's a constant. It's steady. I'm just checking here that the equation is satisfied. The derivative is 0 because it's constant, and f is 0 because it's a critical point. So I have 0 equals 0. The differential equation is perfectly good. So if I start at a critical point, I stay there.

That's not our central question. Our key question is, if I start at other points, do I approach a critical point, or do I go away from it? Is the critical point stable and attractive, or is it unstable and repulsive?

So the way to answer that question is to look at the equation when you're very near the critical point. Very near the critical point, we could make the equation linear. We can linearize the equation, and that's the whole trick. And I've spoken before, and I'll do it again now for one equation. But the real message, the real content comes with two or three equations. That's what we see in nature very often, and we want to know, is the problem stable?

OK. So what does linearize mean? Every function is linear if you look at it through a microscope. Maybe I should say if you blow it up near y equal Y, every function is linear. Here is f of y. Here it's coming through-- it's a graph of f of y, whatever it is. If this we recognize as the point capital Y, right, that's where the function is 0.

And near that point, my function is almost a straight line. And the slope of that tangent is the coefficient, and everything depends on that. Everything depends on whether the slope is going up like that-- probably that's going to be unstable-- or coming down. If it were coming down, then the slope would be negative at the critical point, and probably that will be stable. OK.

So I just have to do a little calculus. The whole idea of linearizing is the central idea of calculus.

That we have curves, but near a point, we can pretend-- they are essentially straight if we focus in, if we zoom in. So this is a zooming-in problem, linearization. OK. So if I zoom in the function at some y. I'm zooming in around the point capital Y. But you remember the tangent line stuff is the function at Y. So little y is some point close by. Capital Y is the crossing point.

And this is the y minus Y times the slope-- that's the slope-- the slope at the critical point there is all that's-- you see that the right-hand side is linear. And actually, f of Y is 0. That's the point. So that I have just a linear approximation with that slope and a simple function. OK.

So I'll use this approximation. I'll put that into the equation, and then I'll have a linear equation, which I can easily solve. Can I do that? So my plan is, take my differential equation, look, focus near the steady state, near the critical point capital Y. Near that point, this is my good approximation to f, and I'll just use it. So I plan to use that right away.

So now here's the linearized. So d by dt of y equals f of y. But I'm going to do approximately equals this y minus capital Y times the slope. So the slope is my coefficient little a in my first-order linear equation. So I'm going back to chapter 1 for this linearization for one equation. But then the next video is the real thing by allowing two equations or even three equations. So we'll make a small start on that, but it's really the next video.

OK. So that's the equation. Now, notice that I could put dy dt as-- the derivative of that constant is 0, so I could safely put it there. So what does this tell me? Let me call that number a. So I can solve that equation, and the solution will be y minus capital Y. It's just linear. The derivative is the thing itself times a.

It's the pure model of steady growth or steady decay. y minus Y is, let's say, some e to the at. Right? When I have a coefficient in the linear equation ay, I see it in the exponential. So a less than 0 is stable. Because a less than 0, that's negative, and the exponential drops to 0. And that tells me that y approaches capital Y. It goes to the critical point, to the steady state, and not away.

Example, example. Let me just take an example that you've seen before, the logistic equation, where the right side is, say, 3y minus y squared. OK. Not linear. So I plan to linearize after I find the critical points. Critical points, this is 0. That equals 0 at-- I guess there will be two critical points because I have a second-degree equation. When that is 0, it could be 0 at y equals 0 or at y equals 3. So two critical points, and each critical point has its own linearization, its slope at that critical point.

So you see, if I graph f of y here, this 3y minus y squared has-- there is 3y minus y squared. There is one critical point, 0. There is the other critical point at 3. Here the slope is positive-- unstable. Here the slope is negative-- stable. So this is stable, unstable.

And let me just push through the numbers here. So the df dy, that's the slope. So I have to take the derivative of that. Notice this is not my differential equation. There is my differential equation. Here is my linearization step, my computation of the derivative, the slope.

So the derivative of that is 3 minus 2y, and I've got two critical points. At capital Y equal 0, that's 3. And at capital Y equals 3, it's 3 minus 6, it's minus 3. Those are the slopes we saw on the picture. Slope up, the parabola is going up. Slope down. So this will correspond to unstable.

So what does it mean for this to be unstable? It means that the solution Y equals 0, constant 0, solves the equation, no problem. If Y stays at 0, it's a perfectly OK solution. The derivative is 0. Everything's 0.

But if I move a little away from 0, if I move a little way from 0, then the 3y minus y squared, what does it look like? If I'm moving just a little away from Y equals 0, away from this unstable point, y squared will be extremely small. So it's really 3y. The y squared will be small near Y equals 0. Forget that. We have exponential growth, e to the 3t. We leave the 0 steady state, and we move on.

Now, eventually we'll move somewhere near the other steady state. At capital Y equals 3, the slope of this thing is minus 3, and the negative one will be the stable point. So where y minus 3, the distance to the steady state, the critical point will grow like e to the mi-- well, will decay, sorry, I said grow, I meant decay-- will decay like e to the minus 3t because the minus 3 in the slope is the minus 3 in the exponent.

OK. That's not rocket science, although it's pretty important for rockets. Let me just say what's coming next and then do it in the follow-up video. So what's coming next will be two equations, dy dt and dz dt. I have two things. y and z, they depend on each other. So the growth or decay of y is given by some function f, and this is given by some different function g, so f and g.

Now, when do I have steady state? When this is 0. When they're both 0. They both have to be 0. And then dy dt is 0, so y is steady. dz dt is 0, so z is steady. So I'm looking for-- I've got two numbers to look for. And I've got two equations, f of y-- oh, let me call that capital Y, capital Z-- so those are numbers now-- equals 0. So I want to solve-- equals 0, and g of capital Y, capital Z equals 0. Yeah, yeah. So both right-hand sides should be 0, and then I'm in a steady state.

But this is going to be like more interesting to linearize. That's really the next video, is how do you linearize? What does the linearized thing look like when you have two functions depending on two variables Y and Z? You're going to have, we'll see, [? for ?] slopes-- well, you'll see it. So this is what's coming. And we end up with a two-by-two matrix because we have two equations, two unknowns, and a little more excitement than the classical single equation, like a logistic equation.

OK. Onward to two.