SOLUTIONS TO CHAPTER 2

2.1 THE DIVERGENCE OPERATOR

2.1.1 From (2.1.5)

$$\text{Div}\mathbf{A} = \frac{\partial(A_x)}{\partial x} + \frac{\partial(A_y)}{\partial y} + \frac{\partial(A_z)}{\partial z}$$

$$= \frac{A_o}{d^2}\left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)\right] \quad (1)$$

$$= \frac{2A_o}{d^2}(x + y + z) \quad (2)$$

2.1.2 (a) From (2.1.5), operating on each vector

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d}\left[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x)\right] = 0 \quad (1)$$

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y)\right] = 0 \quad (2)$$

$$\begin{aligned}\nabla \cdot \mathbf{A} &= A_o\left[\frac{\partial}{\partial x}(e^{-ky}\cos kx) - \frac{\partial}{\partial y}(e^{-ky}\sin kx)\right] \\ &= A_o\left[-ke^{-ky}\sin kx + ke^{-ky}\sin kx\right] = 0\end{aligned} \quad (3)$$

(b) All vectors having only one Cartesian component, a (non-constant) function of the coordinate corresonding to that component. For example, $\mathbf{A} = \mathbf{i}_x f(x)$ or $\mathbf{A} = \mathbf{i}_y g(y)$ where $f(x)$ and $g(y)$ are not constants. The example of Prob. 2.1.1 is a superposition of these possibilities.

2.1.3 From Table I

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad (1)$$

Thus, for (a)

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{A_o}{d}\left[\frac{1}{r}\frac{\partial}{\partial r}(r^2\cos 2\phi) - \frac{\partial}{\partial \phi}(\sin 2\phi)\right] \\ &= \frac{A_o}{d}[2\cos 2\phi - 2\cos 2\phi] = 0\end{aligned} \quad (2)$$

for (b)

$$\nabla \cdot \mathbf{A} = A_o\left[\frac{1}{r}\frac{\partial}{\partial r}r\cos\phi - \frac{1}{r}\frac{\partial}{\partial \phi}\sin\phi\right] = 0 \quad (3)$$

while for (c)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d^2}\frac{1}{r}\frac{\partial}{\partial r}r^3 = \frac{A_o}{d^2}3r \quad (4)$$

2.1.4 From (2),

$$\text{Div}\mathbf{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \mathbf{A} \cdot d\mathbf{s} \tag{1}$$

Following steps like (2.1.3)-(2.1.5)

$$\begin{aligned}\oint_S \mathbf{A} \cdot d\mathbf{a} \simeq &\Delta\phi\Delta z\big[(r + \frac{\delta r}{2})A_r(r + \frac{\delta r}{2}, \phi, z)\big] \\ &- \Delta\phi\Delta z\big[(r - \frac{\Delta r}{2})A_r(r - \frac{\Delta r}{2}, \phi, z)\big] \\ &+ \Delta r\Delta z\big[A_\phi(r, \phi + \frac{\Delta\phi}{2}, z) - A_\phi(r, \phi - \frac{\Delta\phi}{2}, z)\big] \\ &+ r\Delta\phi\Delta r\big[A_z(r, \phi, z + \frac{\Delta z}{2}) - A_z(r, \phi, z - \frac{\Delta z}{2})\big]\end{aligned} \tag{2}$$

Thus, the limit

$$\begin{aligned}\text{Div}\mathbf{A} = &\lim_{r\Delta\phi\Delta z \to 0} \\ &\Big\{r\Delta\phi\Delta z\frac{\big[(r + \Delta r)A_r(r + \frac{\Delta r}{2}, \phi, z) - (r - \frac{\Delta r}{2})A_r(r - \frac{\Delta r}{2}, \phi, z)\big]}{r\Delta\phi\Delta z\Delta r} \\ &+ \frac{\big[A_\phi(r, \phi + \frac{\Delta\phi}{2}, z) - A_\phi(r, \phi - \frac{\Delta\phi}{2}, z)\big]}{r\Delta\phi} \\ &+ \frac{\big[A_z(r, \phi, z + \frac{\Delta z}{2}) - A_z(r, \phi, z - \frac{\Delta z}{2})\big]}{\Delta z}\Big\}\end{aligned} \tag{3}$$

gives the result summarized in Table I.

2.1.5 From Table I,

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{1}$$

For (a)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d^3}\big[\frac{1}{r^2}\frac{\partial}{\partial r}(r^5)\big] = \frac{A_o}{d^3}(5r^2) \tag{2}$$

for (b)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2) = 0 \tag{3}$$

and for (c)

$$\begin{aligned}\nabla \cdot \mathbf{A} &= A_o\big[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cos\theta) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\big] \\ &= A_o\big[\frac{2}{r}\cos\theta - \frac{2\sin\theta\cos\theta}{r\sin\theta}\big] = 0\end{aligned} \tag{4}$$

2.1.6 Starting with (2) and using the volume element shown in Fig. S2.1.6,

$$\oint_S \mathbf{A}\cdot d\mathbf{a} = \lim_{(\Delta r)(r\Delta\theta)(r\sin\theta\Delta\phi)\to 0}\Big\{(r+\frac{\Delta r}{2})\Delta\theta(r+\frac{\Delta r}{2})\sin\theta\Delta\phi A_r(r+\frac{\Delta r}{2},\theta,\phi)$$
$$-(r-\frac{\Delta r}{2})\Delta\theta(r-\frac{\Delta r}{2})\sin\theta\Delta\phi A_r(r-\frac{\Delta r}{2},\theta,\phi)$$
$$+\Delta r r\Delta\phi\big[\sin(\theta+\frac{\Delta\theta}{2})A_\phi(r,\theta+\frac{\Delta\theta}{2},\phi)$$
$$-\sin(\theta-\frac{\Delta\theta}{2})A_\phi(r,\theta-\frac{\Delta\theta}{2},\phi)\big]$$
$$+r\Delta\theta\Delta r\big[A_\phi(r,\theta,\phi+\frac{\Delta\phi}{2})-A_\phi(r,\theta,\phi-\frac{\Delta\phi}{2})\big]\Big\} \quad (1)$$

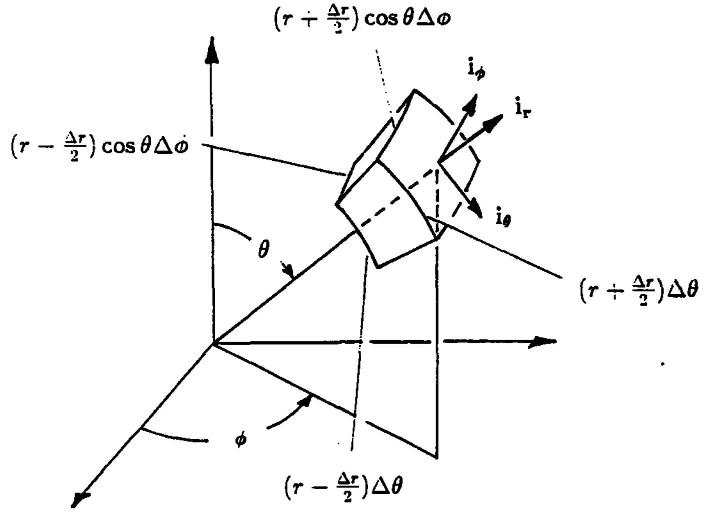


Figure S2.1.6

Thus,

$$\nabla\cdot\mathbf{A} = \frac{\oint_S \mathbf{A}\cdot d\mathbf{a}}{(\Delta r)(r\Delta\theta)(r\sin\theta\Delta\phi)}$$
$$= \lim_{\Delta r\to 0}\Big\{\frac{1}{r^2}\frac{\big[(r+\frac{\Delta r}{2})^2A_r(r+\frac{\Delta r}{2},\theta,\phi)-(r-\frac{\Delta r}{2})^2A_r(r-\frac{\Delta r}{2},\theta,\phi)\big]}{\Delta r}$$
$$+\lim_{\Delta\theta\to 0}\sin\theta\frac{1}{r}\frac{\big[\sin(\theta+\frac{\Delta\theta}{2})A_\theta(r,\theta+\frac{\Delta\theta}{2},\phi)-\sin(\theta-\frac{\Delta\theta}{2})A_\theta(r,\theta-\frac{\Delta\theta}{2},\phi)\big]}{\Delta\theta}$$
$$+\lim_{\Delta\phi\to 0}\frac{1}{r\sin\theta}\frac{\big[A_\phi(r,\theta,\phi+\frac{\Delta\phi}{2})-A_\phi(r,\theta,\phi-\frac{\Delta\phi}{2})\big]}{\Delta\phi}\Big\} \quad (2)$$

In the limit

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{3}$$

2.2 GAUSS' INTEGRAL THEOREM

2.2.1

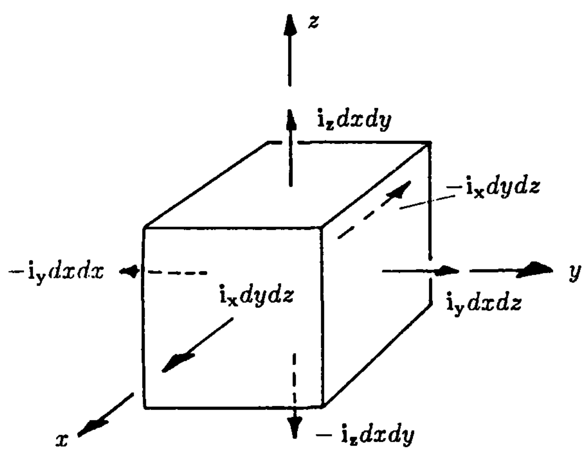


Figure S2.2.1

(a) The vector surface elements are shown in Fig. S2.2.1.

(b) There is no z contribution, so there are only $x = \pm d$ surfaces, $A_x = (A_o/d)(\pm d)$ and $\mathbf{n} = \pm\mathbf{i}_x dydz$. Hence, the first two integrals. The second and third are similar.

(c) From (2.1.5)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d}\left[\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right] = \frac{2A_o}{d} \tag{1}$$

Thus, because $\nabla \cdot \mathbf{A}$ is constant over the volume

$$\int_V \nabla \cdot \mathbf{A} dV = \frac{2A_o}{d}(2d)^3 = 16A_o d^2 \tag{2}$$

2.2.2 The surface integration is

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \frac{A_o}{d^3}\left[\int_{-d}^{d}\int_{-d}^{d} dy^2 dydz - \int_{-d}^{d}\int_{-d}^{d}(-d)y^2 dydz + \int_{-d}^{d}\int_{-d}^{d} dx^2 dxdz - \int_{-d}^{d}\int_{-d}^{d}(-d)x^2 dxdz \right. \tag{1}$$

From the first integral

$$= \frac{A_o}{d^3}(2d^2)\left(\frac{2}{3}d^3\right) \tag{2}$$

The others give the same contribution, so

$$= \frac{4A_o}{d^3}\frac{4d^5}{3} = \frac{16A_o d^2}{3} \tag{3}$$

To evaluate the right hand side of (2.2.4)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d^3}\left[\frac{\partial}{\partial x}xy^2 + \frac{\partial}{\partial y}x^2 y\right] = \frac{A_o}{d^3}(y^2 + x^2) \tag{4}$$

So, indeed

$$\int_V \nabla \cdot \mathbf{A} dv = \int_{-d}^{d}\int_{-d}^{d}\int_{-d}^{d} \frac{A_o}{d^3}(y^2 + x^2)dydxdz = \frac{16}{3}A_o d^2 \tag{5}$$

2.3 GAUSS' LAW, MAGNETIC FLUX CONTINUITY AND CHARGE CONSERVATION

2.3.1 (a) From Prob. 1.3.1

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}\left[\frac{x}{x^2+y^2}\mathbf{i_x} + \frac{y}{x^2+y^2}\mathbf{i_y}\right] \tag{1}$$

From (2.1.5)

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\lambda}{2\pi\epsilon_o}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right] \\
&= \frac{\lambda}{2\pi\epsilon_o}\left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right] \\
&= \frac{\lambda}{2\pi\epsilon_o}\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] = 0
\end{aligned} \tag{2}$$

except where $x^2 + y^2 = 0$ (on the z-axis).

(b) In cylindrical coordinates

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}\frac{1}{r}\mathbf{i_r} \tag{3}$$

Thus, from Table I,

$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\lambda}{2\pi\epsilon_o}\right) = 0 \tag{4}$$

2.3.2 From Table I in cylindrical coordinates with $\partial(\;)/\partial\phi$ and $\partial(\;)/\partial z = 0$,

$$\nabla \cdot \epsilon_o \mathbf{E} = \frac{\epsilon_o}{r}\frac{\partial}{\partial r}(rE_r) \tag{1}$$

so

$$\nabla \cdot \epsilon_o \mathbf{E} = \frac{\epsilon_o}{r}\frac{\partial}{\partial r}\begin{cases} \rho_o r^4/4\epsilon_o b^2; & r < b \\ \rho_o b^2/4\epsilon_o; & b < r < a \end{cases} \tag{2}$$

$$= \begin{cases} \rho_o r^2/b^2; & r < b \\ 0; & b < r < a \end{cases} \tag{3}$$

2.3.3 Using $\mathbf{H} = H_o(\mathbf{i_x} + \mathbf{i_y})$ in (2.1.5),

$$\nabla \cdot \mu_o \mathbf{H} = \mu_o H_o\left[\frac{\partial(1)}{\partial x} + \frac{\partial(1)}{\partial y}\right] = 0 \tag{1}$$

2.3.4 In cylindrical coordinates (Table I):

$$\nabla \cdot \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{1}{r}\frac{\partial H_\phi}{\partial \phi} + \frac{\partial H_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{i}{2\pi r}\right) = 0 \tag{1}$$

2.3.5 If $\nabla \cdot \mu_o \mathbf{H} = 0$ *everywhere* then the integral of its normal over an arbitrary closed surface in that region will be zero and

(a)

$$\nabla \mu_o \mathbf{H} = 0$$

(b)

$$\nabla \cdot \mu_o \mathbf{H} = \frac{H_o}{a}\frac{\partial x}{\partial x} = \frac{H_o}{a}$$

(c)

$$\nabla \cdot \mu_o \mathbf{H} = \frac{H_o}{a}\frac{\partial y}{\partial x} = 0$$

Thus, only (b) will not satisfy (1.7.1)

2.3.6 Evaluation using (2.1.5) gives

$$\rho = \nabla \cdot \epsilon_o \mathbf{E} = \epsilon_o \frac{\partial E_z}{\partial z} = \frac{2\rho_o}{s} z$$

which is the given charge density.

2.3.7 Using $\nabla \cdot \mathbf{F}$ in spherical coordinates from Table I with $\partial/\partial\theta$ and $\partial/\partial\phi = 0$,

$$\nabla \cdot \mathbf{J} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{3}\frac{d\rho_o}{dt}\right) = -\frac{d\rho_o}{dt}$$

which, since ρ_o is independent of r, checks with (2.3.3).

2.4 THE CURL OPERATOR

2.4.1 All cases have only x and y components, independent of z.

$$\nabla \times \mathbf{A} = \begin{bmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ A_x & A_y & 0 \end{bmatrix} = \mathbf{i_z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

Thus
(a)

$$\nabla \times \mathbf{A} = \frac{A_o}{d}[1-1] = 0 \tag{1}$$

(b)

$$\nabla \times \mathbf{A} = \frac{A_o}{d}[0-0] = 0 \tag{2}$$

(c)

$$\nabla \times \mathbf{A} = A_o[-e^{-ky}\cos kx + ke^{-ky}\cos kx] = 0 \tag{3}$$

To make a finite curl make a single component having any dependence on a coordinate perpendicular to the vector.

$$A_y = f(x), \qquad A_x = 0,\ A_z = 0 \tag{4}$$

Say,

$$f(x) = x, x^2, x^3 \Rightarrow \nabla \times \mathbf{A} = \frac{\partial f(x)}{\partial x} = 1, 2x, 3x^2 \tag{5}$$

2.4.2 In all cases $A_z = 0$ and $\partial/\partial z = 0$, so from Table I,

$$\nabla \times \mathbf{A} = \mathbf{i_z}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right] \tag{1}$$

(a) Thus

$$\begin{aligned}(a) \Rightarrow \nabla \times \mathbf{A} &= \mathbf{i_z}\frac{A_o}{d}\left[\frac{1}{r}\frac{\partial}{\partial r}(-r^2\sin 2\phi) - \frac{1}{r}\frac{\partial}{\partial \phi}(r\cos 2\phi)\right] \\ &= \mathbf{i_z}\frac{A_o}{d}[-2\sin 2\phi + 2\sin 2\phi] = 0\end{aligned} \tag{2}$$

$$(b) \Rightarrow \nabla \times \mathbf{A} = \mathbf{i}_z A_o\left[\frac{1}{r}\frac{\partial}{\partial r}(-r\sin\phi) - \frac{1}{r}\frac{\partial}{\partial \phi}\cos\phi\right] = \mathbf{i}_z A_o\left[-\frac{\sin\phi}{r} + \frac{\sin\phi}{r}\right] = 0 \tag{3}$$

$$(c) \Rightarrow \nabla \times \mathbf{A} = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{A_o r^3}{d^2}\right) = \mathbf{i}_z\left(\frac{3A_o r}{d^2}\right) \tag{4}$$

(b) Possible vector functions having a curl make $\mathbf{A} = A_\phi \mathbf{i}_\phi$ where $rA_\phi = f(r)$ is not a constant. For example $f(r) = r, r^2, r^3$, in which case

$$\nabla \times \mathbf{A} = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}(r^n) = nr^{n-2} \tag{5}$$

2.4.3 From (2)

$$(\text{curl}\mathbf{A})_n = \lim_{\Delta a \to 0} \frac{1}{\Delta a}\oint_C \mathbf{A}\cdot d\mathbf{s} \tag{1}$$

Using contour of Fig. P2.4.3a,

$$(\nabla \times \mathbf{A})_r = \lim_{r\Delta\phi\Delta z \to 0}\left\{\frac{\left[\Delta z A_z(r,\phi+\frac{\Delta\phi}{2},z) - \Delta z A_z(r,\phi-\frac{\Delta\phi}{2},z)\right]}{r\Delta\phi\Delta z} - \frac{\left[r\Delta\phi A_\phi(r,\phi,z+\frac{\Delta z}{2}) - r\Delta\phi A_\phi(r,\phi,z-\frac{\Delta z}{2})\right]}{r\Delta\phi\Delta z}\right\} = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \tag{2}$$

Using the contour of Fig. P2.4.3b

$$(\nabla \times \mathbf{A})_\phi = \lim_{\Delta r\Delta z \to 0}\left\{\frac{\left[\Delta r A_r(r,\phi,z+\frac{\Delta z}{2}) - \Delta r A_r(r,\phi,z-\frac{\Delta z}{2})\right]}{\Delta r\Delta z} - \frac{\left[\Delta z A_z(r+\frac{\Delta r}{2},\phi,z) - \Delta z A_z(r-\frac{\Delta r}{2},\phi,z)\right]}{\Delta r\Delta z}\right\} = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \tag{3}$$

$$(\nabla \times \mathbf{A})_z = \lim_{\Delta r r\Delta\phi \to 0}\left\{\frac{\left[(r+\frac{\Delta r}{2})\Delta\phi A_\phi(r+\frac{\Delta r}{2},\phi,z) - (r-\frac{\Delta r}{2})\Delta\phi A_\phi(r-\frac{\Delta r}{2},\phi,z)\right]}{\Delta r r\Delta\phi} - \frac{\left[\Delta r A_r(r,\phi+\frac{\Delta\phi}{2},z) - \Delta r A_r(r,\phi-\frac{\Delta\phi}{2},z)\right]}{\Delta r r\Delta\phi}\right\} = \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi} \tag{4}$$

2.4.4

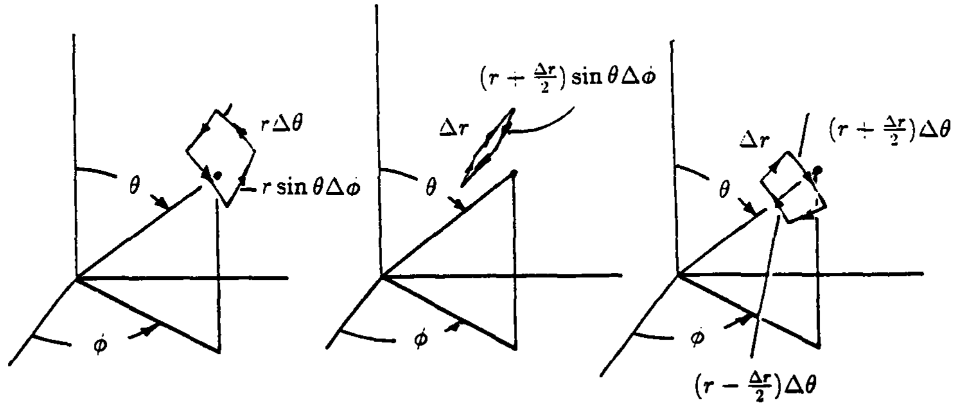


Figure S2.4.4

From (2)

$$(\nabla\times\mathbf{A})_r = \lim_{r\Delta\theta r\sin\theta\Delta\phi\to 0}\left\{-\frac{\left[r\Delta\theta A_\theta\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right) - r\Delta\theta A_\theta\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\right]}{r\Delta\theta r\sin\theta\Delta\phi}\right.$$
$$\left.+\frac{\left[r\sin\left(\theta+\frac{\Delta\theta}{2}\right)\Delta\phi A_\phi\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) - r\sin\left(\theta-\frac{\Delta\theta}{2}\right)\Delta\phi A_\phi\left(r,\theta-\frac{\Delta\theta}{2},\phi\right)\right]}{r\Delta\theta r\sin\theta\Delta\phi}\right\}$$
$$= -\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} \tag{1}$$

$$(\nabla\times\mathbf{A})_\theta = \lim_{\Delta r\sin\theta r\Delta\phi\to 0}\left\{\frac{\left[\Delta r A_r\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right) - \Delta r A_r\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\right]}{\Delta r\sin\theta r\Delta\phi}\right.$$
$$\left.-\frac{\left[\Delta\phi\sin\theta\left(r+\frac{\Delta r}{2}\right)A_\phi\left(r+\frac{\Delta r}{2},\theta,\phi\right) - \Delta\phi\sin\phi\left(r-\frac{\Delta r}{2}\right)A_\phi\left(r-\frac{\Delta r}{2},\theta,\phi\right)\right]}{\Delta r\sin\theta r\Delta\phi}\right\}$$
$$= \frac{1}{r(\sin\theta)}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} \tag{2}$$

$$(\nabla\times\mathbf{A})_\phi = \lim_{r\Delta\theta\Delta r\to 0}\left\{\frac{\left[\Delta\theta\left(r+\frac{\Delta r}{2}\right)A_\theta\left(r+\frac{\Delta r}{2},\theta,\phi\right) - \Delta\theta\left(r-\frac{\Delta r}{2}\right)A_\theta\left(r-\frac{\Delta r}{2},\theta,\phi\right)\right]}{r\Delta\theta\Delta r}\right.$$
$$\left.-\frac{\left[\Delta r A_r\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) - \Delta r A_r\left(r,\theta-\frac{\Delta\theta}{2},\phi\right)\right]}{r\Delta\theta\Delta r}\right\}$$
$$= \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta} \tag{3}$$

2.4.5 (a) Stokes' integral theorem, (2.4.1) is

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} \tag{1}$$

With S a closed surface, $C \to 0$, so

$$\oint_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = 0 = \int_V \nabla \cdot (\nabla \times \mathbf{A}) dV \tag{2}$$

Because V is arbitrary, the integrand of this volume integral must be zero.

(b) Carrying out the operations gives

$$\nabla \cdot (\nabla \times \mathbf{A}) = \frac{\partial}{\partial x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] = 0 \tag{3}$$

2.5 STOKES' INTEGRAL THEOREM

2.5.1

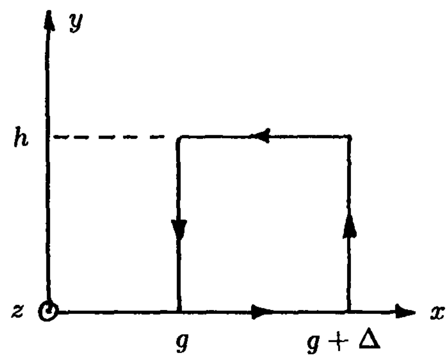


Figure S2.5.1

(a) Using Fig. S2.5.1 to construct $\mathbf{A} \cdot d\mathbf{s}$,

$$\begin{aligned}\oint_C \mathbf{A} \cdot d\mathbf{s} &= \int_g^{g+\Delta} A_x(x,0)dx + \int_0^h A_y(g+\Delta, y)dy \\ &- \int_g^{g+\Delta} A_x(x,h)dx - \int_0^h A_y(g,y)dy \\ &= \int_g^{g+\Delta} (0)dx + \int_0^h \frac{A_o}{d^2}(g+\Delta)^2 dy \\ &- \int_g^{g+\Delta} (0)dx - \int_0^h \frac{A_o}{d^2} g^2 dy \\ &= \frac{A_o}{d^2}[(g+\Delta)^2 h - g^2 h]\end{aligned} \tag{1}$$

(b) The integrand of the surface integral is

$$\nabla\times\mathbf{A} = \begin{bmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \partial/\partial x & 0 & 0 \\ 0 & A_y & 0 \end{bmatrix} = \mathbf{i_z}\frac{\partial A_y}{\partial x} = \mathbf{i_z}\frac{2A_o x}{d^2}$$

Thus

$$\int_S \nabla\times\mathbf{A}\cdot d\mathbf{a} = \int_0^h\int_g^{g+\Delta}\frac{2A_o x}{d^2}dxdy = \frac{A_o}{d^2}[(g+\Delta)^2 - g^2]h \qquad (2)$$

2.5.2 (a) Using the contour shown in Fig. S2.5.1,

$$\begin{aligned}\oint_C \mathbf{A}\cdot d\mathbf{s} &= \frac{A_o}{d}[\int_g^{g+\Delta}(0)dx + \int_0^h(g+\Delta)dy \\ &\quad - \int_g^{g+\Delta}(-h)dx - \int_0^h g dy] \\ &= \frac{A_o}{d}[(g+\Delta)h + h\Delta - gh] = \frac{2A_o h\Delta}{d}\end{aligned} \qquad (1)$$

(b) To get the same result carrying out the surface integral,

$$\begin{aligned}\nabla\times\mathbf{A} &= \begin{bmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \partial/\partial x & \partial/\partial y & 0 \\ A_x & A_y & 0 \end{bmatrix} = \mathbf{i_z}[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}] \\ &= \frac{A_o}{d}[1+1] = \frac{2A_o}{d}\end{aligned}$$

and hence

$$\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \frac{2A_o}{d}(\Delta h) \qquad (2)$$

2.6 DIFFERENTIAL LAWS OF AMPERE AND FARADAY

2.6.1 From Prob. 1.4.2

$$\mathbf{H} = \frac{J_o}{2}\begin{cases} -y\mathbf{i_x} + x\mathbf{i_y}; & r < b \\ -b^2 y(x^2+y^2)^{-1}\mathbf{i_x} + b^2 x(x^2+y^2)^{-1}\mathbf{i_y}; & b < r < a \end{cases} \qquad (1)$$

Thus,

$$\begin{aligned}\nabla\times\mathbf{H} &= \mathbf{i_z}[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}] \\ &= \frac{J_o}{2}\begin{cases} 1-(-1) = 2 & r < b \\ \frac{b^2}{x^2+y^2} - \frac{2b^2x^2}{(x^2+y^2)^2} - \frac{(-b^2)}{(x^2+y^2)} + \frac{2(-b^2)y^2}{(x^2+y^2)} = 0; & b < r < a \end{cases}\end{aligned} \qquad (2)$$

Thus, $\nabla\times\mathbf{H} = \mathbf{J}$ at each point, r.

2.6.2 Ampére's differential law is written in cylindrical coordinates using the expression for $\nabla \times \mathbf{H}$ from Table I with $\partial/\partial\phi$ and $\partial/\partial z = 0$ and $H_r = 0$, $H_z = 0$. Thus

$$\nabla \times \mathbf{H} = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}\{J_o a^2\left[1 - e^{-r/a}\left(1 + \frac{r}{a}\right)\right]\} = J_o e^{-r/a}\mathbf{i}_z \tag{1}$$

2.7 VISUALIZATION OF FIELDS AND THE DIVERGENCE AND CURL

2.7.1 (a) For ρ and $\mathbf{E}$ given by

$$\rho = \frac{2\rho_o z}{s}$$

$$E_z = \frac{\rho_o}{\epsilon_o s}\left[z^2 - \left(\frac{s}{2}\right)^2\right] \tag{1}$$

the sketch is shown in Fig. S2.7.1

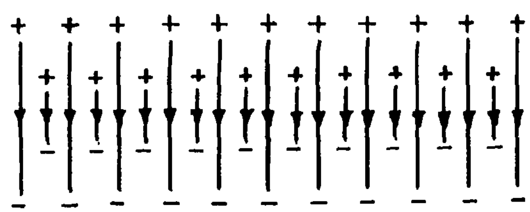


Figure S2.7.1

(b)

$$\nabla \times \mathbf{E} = \begin{bmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ 0 & 0 & \partial/\partial z \\ 0 & 0 & E_z \end{bmatrix} = 0 \tag{2}$$

(c) The density of field lines does not vary in the direction perpendicular to lines.

2.7.2 (a) From Prob. 1.4.1,

$$J_z = J_o e^{-r/a}; \quad H_\phi = \frac{J_o a^2}{r}\left[1 - e^{-r/a}\left(1 + \frac{r}{a}\right)\right] \tag{1}$$

and the field and current plot is as shown in cross-section by Fig. S2.7.2.

(b) From Prob. 1.4.4, the currents are a line current at the origin returned as two surface currents.

$$K_z = \begin{cases} I/\pi(2a+b); & r = a \\ \frac{1}{2}I/\pi(2a+b); & r = b \end{cases} \tag{2}$$

In the annular regions,

$$H_\phi = -\frac{I}{2\pi}\begin{cases} 1/r; & 0 < r < b \\ 2a/r(2a+b); & b < r < a \end{cases} \tag{3}$$

This distribution of current density and magnetic field intensity is shown in cross-section by Fig. S2.7.2.

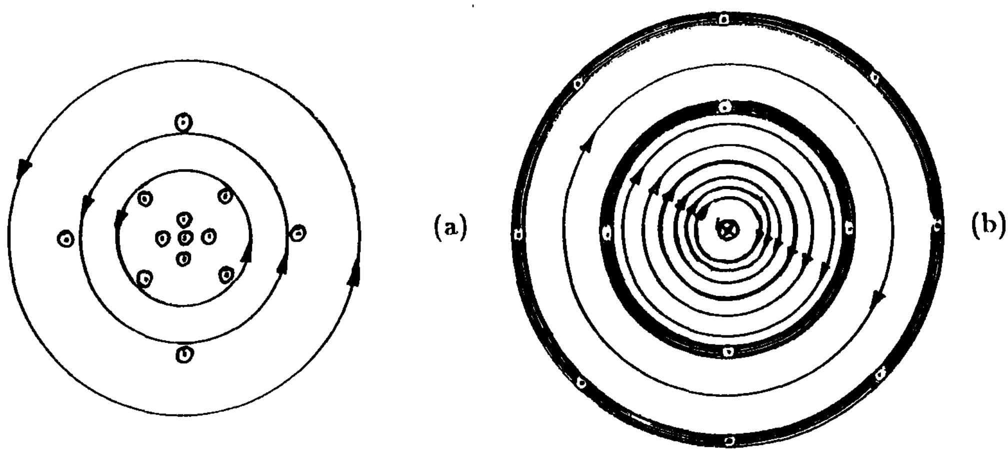


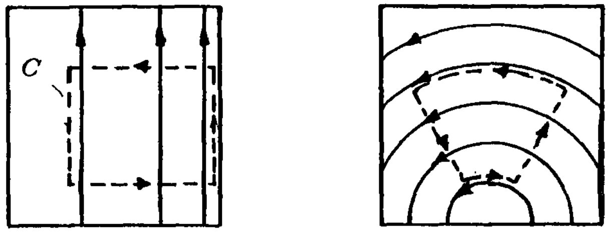
Figure S2.7.2

(c) Because **H** has no ϕ dependence with its only component in the ϕ direction, it must be solenoidal. To check that this is so, note that $\partial/\partial\phi = 0$ and $\partial/\partial z = 0$ and that (from Table I)

$$\nabla \cdot \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_r) = 0 \tag{4}$$

(d) See (c).

2.7.3 (a) The only irrotational field is (b), where the lines are uniform in the direction perpendicular to their direction. In (a), the line integral of the field around a contour such as that shown in Fig. S2.7.3a must be finite. Similarly, because the field intensity is independent of radius in case (c), the line integral shown in Fig. S2.7.3b must be finite.



(a) (b)

Figure S2.7.3

2.7.4 The respective fields are

$$\mathbf{E} = \frac{\sigma_o}{\epsilon_o}\mathbf{i_x} + \frac{2\sigma_o}{\epsilon_o}\mathbf{i_y} \tag{1}$$

$$\mathbf{E} = \frac{\sigma_o}{\epsilon_o}\mathbf{i_x} + \frac{\sigma_o}{\epsilon_o}\mathbf{i_y} \tag{2}$$

and the field plot is as shown in Fig. S2.7.4. Note that the spacing between lines is lesser above to reflect the greater intensity of the field there.

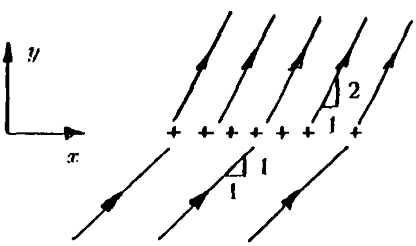


Figure S2.7.4

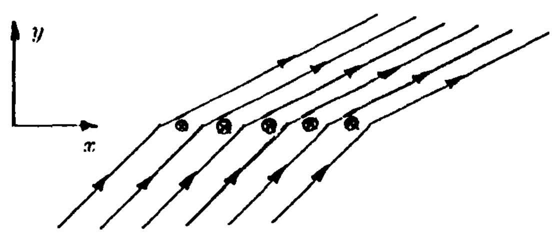


Figure S2.7.5

2.7.5 The respective fields are

$$\mathbf{H} = K_o\mathbf{i_y} + 2K_o\mathbf{i_z} \tag{1}$$

$$\mathbf{H} = K_o\mathbf{i_y} + K_o\mathbf{i_z} \tag{2}$$

and the field plot is as shown in Fig. S2.7.5. Note that, because the field is solenoidal, the number of field lines above and below can be the same while having their spacing reflect the field intensity.

2.7.6 (a) The tangential **E** must be continuous, as shown in Fig. S2.7.6a, so the normal **E** on top must be larger. Because there is than a net flux of **E** out of the interface, it follows from Gauss' integral law [continuity condition (1.3.17)] that the surface charge density is positive.

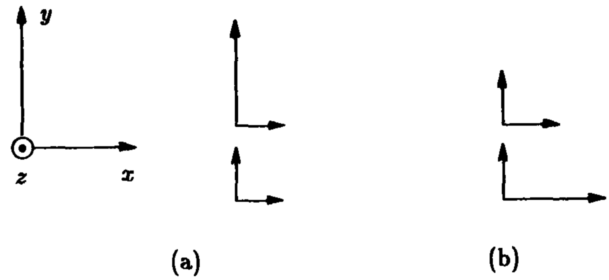


Figure S2.7.6

(b) The normal component of the flux density $\mu_o\mathbf{H}$ is continuous, as shown in Fig. S2.7.6b, so the tangential component on the bottom is largest. From Ampére's integral law [the continuity condition (1.4.16)] it follows that $K_z > 0$.