

By now, we have defined the notion of independence of events and also the notion of independence of random variables.

The two definitions look fairly similar, but the details are not exactly the same, because the two definitions refer to different situations.

For two events, we know what it means for them to be independent.

The probability of their intersection is the product of their individual probabilities.

Now, to make a relation with random variables, we introduce the so-called indicator random variables.

So for example, the random variable  $X$  is defined to be equal to 1 if event  $A$  occurs and to be equal to 0 if event  $[A]$  does not occur.

And there is a similar definition for random variable  $Y$ .

In particular, the probability that random variable  $X$  takes the value of 1, this is the probability that event  $A$  occurs.

It turns out that the independence of the two events,  $A$  and  $B$ , is equivalent to the independence of the two indicator random variables.

And there is a similar statement, which is true more generally.

That is,  $n$  events are independent if and only if the associated  $n$  indicator random variables are independent.

This is a useful statement, because it allows us to sometimes, instead of manipulating events, to manipulate random variables, and vice versa.

And depending on the context, one maybe easier than the other.

Now, the intuitive content is that events  $A$  and  $B$  are independent if the occurrence of event  $A$  does not change your beliefs about  $B$ . And in terms of random variables, one random variable taking a certain value, which indicates whether event  $A$  has occurred or not does not give you any information about the other random variable, which would tell you whether event  $B$  has occurred or not.

It is instructive now to go through the derivation of this fact, at least for the case of two events, because it gives us perhaps some additional understanding about the precise content of the definitions we have introduced.

So let us suppose that random variables  $X$  and  $Y$  are independent.

What does that mean?

Independence means that the joint PMF of the two random variables,  $X$  and  $Y$ , factors as a product of the corresponding marginal PMFs.

And this factorization must be true no matter what arguments we use inside the joint PMF.

And the combination of  $X$  and  $Y$  in this instance have a total of four possible values.

These are the combinations of zeroes and ones that we can form.

And for this reason, we have a total of four equations.

These four equalities are what is required for  $X$  and  $Y$  to be independent.

So suppose that this is true, that the random variables are independent.

Let us take this first relation and write it in probability notation.

The random variable  $X$  taking the value of 1, that's the same as event  $A$  occurring.

And random variable  $Y$  taking the value of 1, that's the same as event  $B$  occurring.

So the joint PMF evaluated at 1, 1 is the probability that events  $A$  and  $B$  both occur.

On the other side of the equation, we have the probability that  $X$  is equal to 1, which is the probability that  $A$  occurs, and similarly, the probability that  $B$  occurs.

But if this is true, then by definition,  $A$  and  $B$  are independent events.

So we have verified one direction of this statement.

If the random variables are independent, then events  $A$  and  $B$  are independent.

Now, we would like to verify the reverse statement.

So suppose that events  $A$  and  $B$  are independent.

In that case, this relation is true.

And as we just argued, this relation is the same as this relation but just written in different notation.

So we have shown that if  $A$  and  $B$  are independent, this relation will be true.

But how about the remaining three relations?

We have more work to do.

Here's how we can proceed.

If A and B are independent, we have shown some time ago that events A and B complement will also be independent.

Intuitively, A doesn't tell you anything about B occurring or not.

So A does not tell you anything about whether B complement will occur or not.

Now, these two events being independent, by the definition of independence, we have that the probability of A intersection with B complement is the product of the probabilities of A and of B complement.

And then we realize that this equality, if written in PMF notation, corresponds exactly to this equation here.

Event A corresponds to X taking the value of 1, event B complement corresponds to the event that Y takes the value of 0.

By a similar argument, B and A complement will be independent.

And we translate that into probability notation.

And then we translate this equality into PMF notation.

And we get this relation.

Finally, using the same property that we used to do the first step here, we have that A complement and B complement are also independent.

And by following the same line of reasoning, this implies the fourth relation as well.

So we have verified that if events A and B are independent, then we can argue that all of these four equations will be true.

And therefore, random variables X and Y will also be independent.