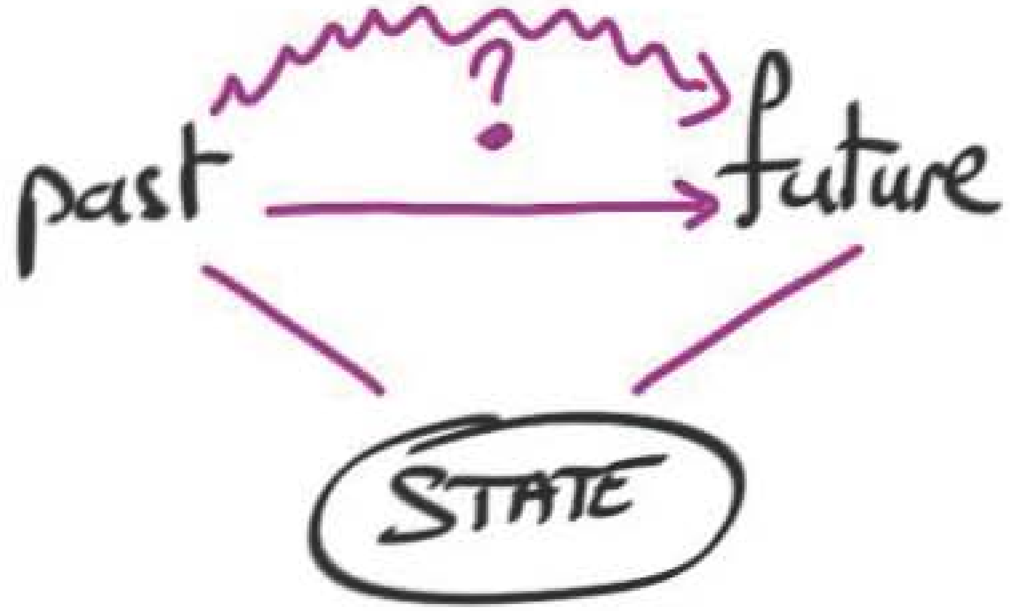
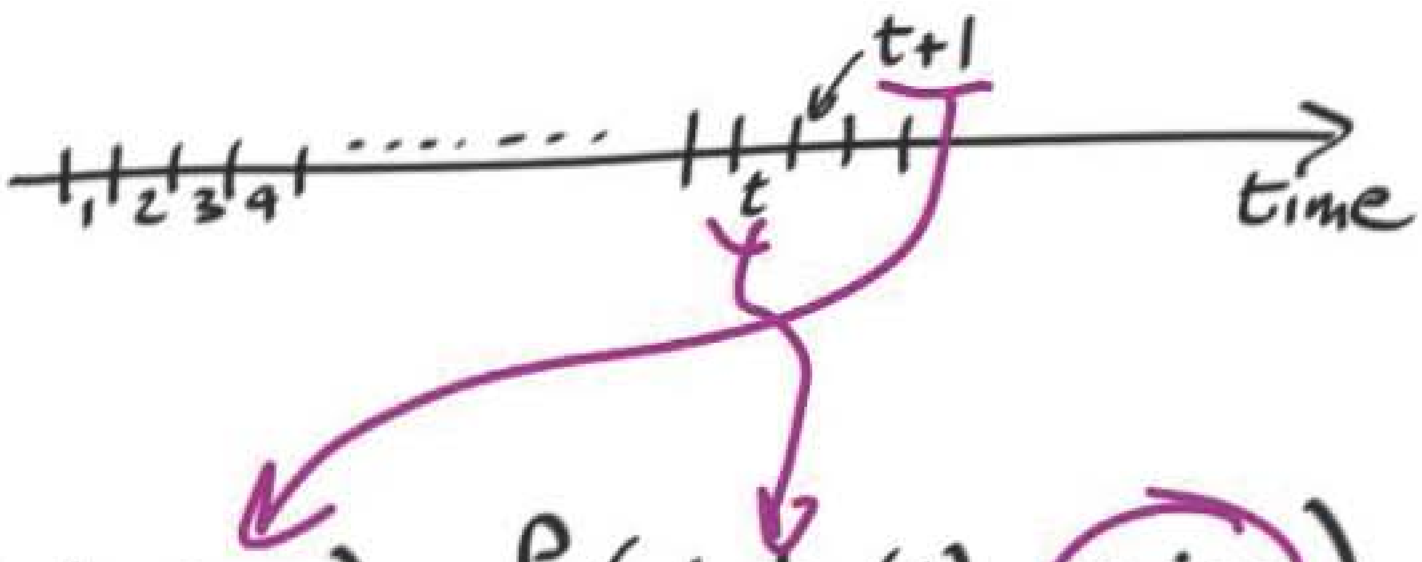
Markov processes – I

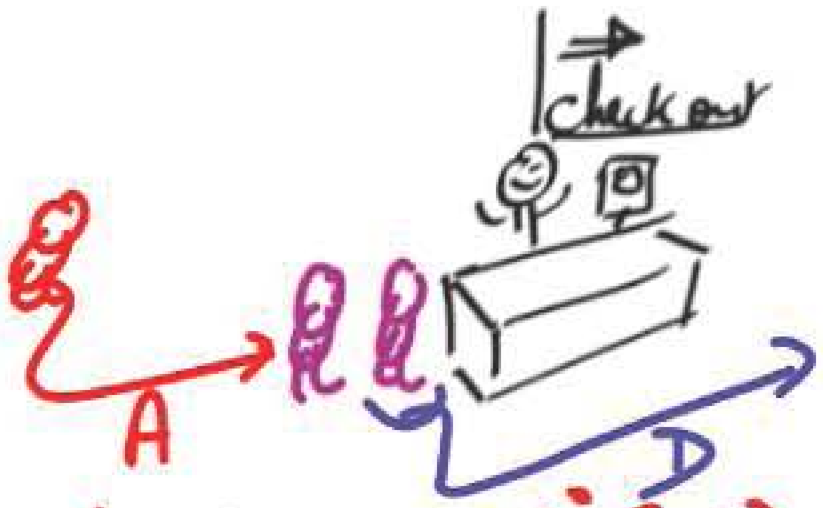
- checkout counter example ✓
- Markov process definition ✓
- n-step transition probabilities ✓
- classification of states ✓





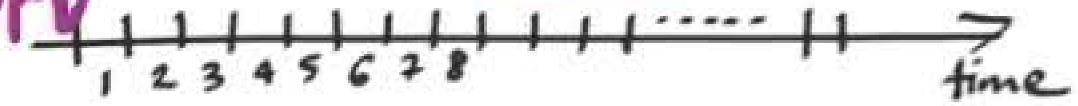
$$\text{state}(t+1) = f(\text{state}(t), \text{noise})$$

checkout counter example



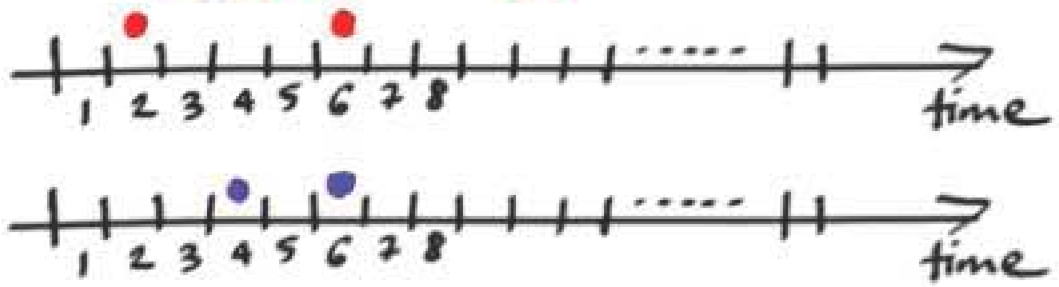
- discrete time $n = 0, 1, \ldots$

6:45pm



- customer arrivals: Bernouilli (p)
- customer service times: geometric(q)
- "state" X_n: number of customers at time n

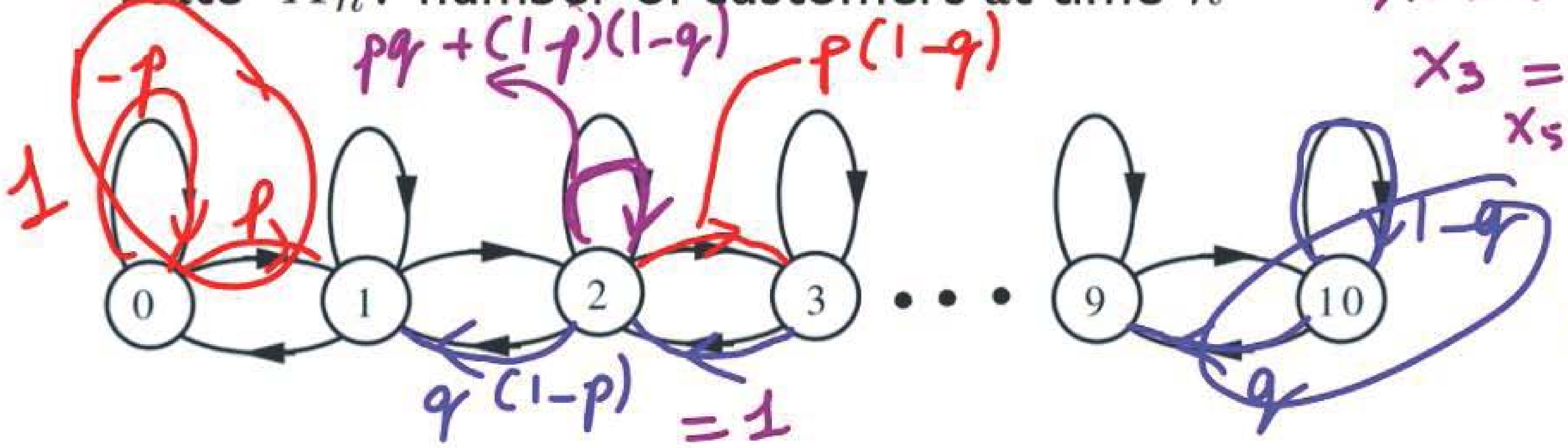
4 ← geometric(p)



$X_0 = 2$ $X_1 = 2$ $X_2 = 2+1 = 3$

$X_3 = 3$ $X_4 = 3-1 = 2$

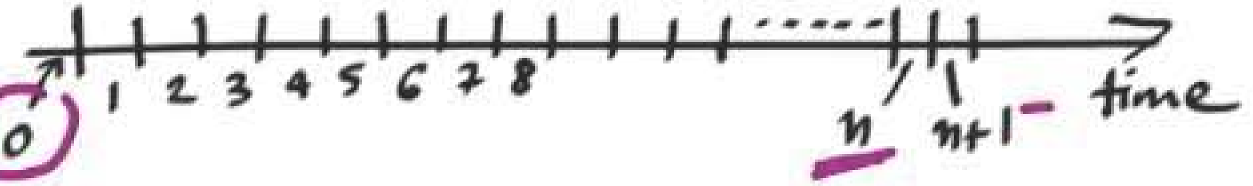
$X_5 = 2$ $X_6 = 2+1-1 = 2, \ldots$



$= 1$ } transition probability graph

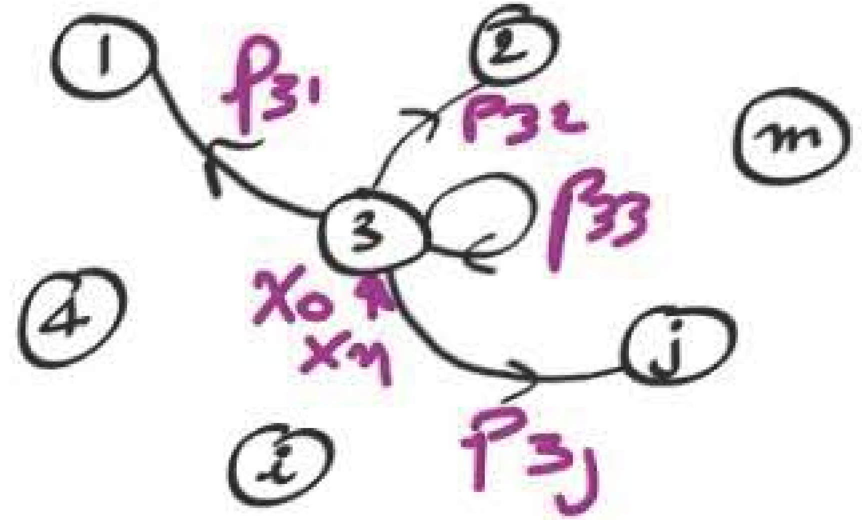
discrete-time finite state Markov chains

"at time n"



- X_n state after n transitions
  - belongs to a finite set
  - initial state X_0 either given or random
  - transition probabilities:

$p_{31} + p_{32} + p_{33} + p_{3j} = 1$



$$
\begin{aligned}
p_{ij} &= \mathbf{P}(X_1 = j \mid X_0 = i) \quad \forall n \\
&= \mathbf{P}(X_{n+1} = j \mid X_n = i)
\end{aligned}
$$

time homogeneous

$\sum_j p_{ij} = 1$

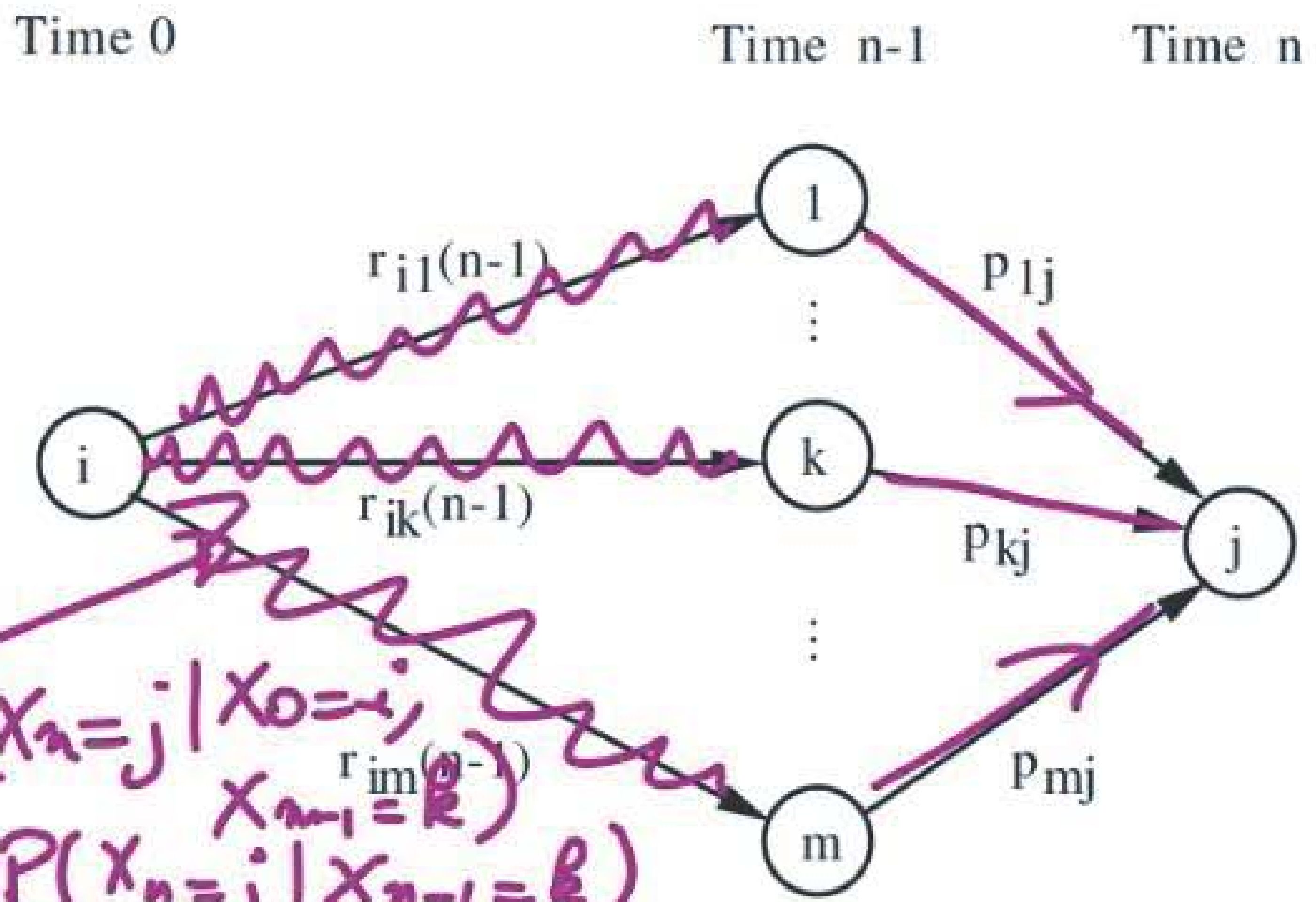
- Markov property/assumption:
  "given current state, the past doesn't matter"

$$
\begin{aligned}
p_{ij} &\overset{\text{def}}{=} \mathbf{P}(X_{n+1} = j \mid X_n = i) \\
&= \mathbf{P}(X_{n+1} = j \mid X_n = i,\; X_{n-1}, \ldots, X_0)
\end{aligned}
$$

- model specification: identify states, transitions, and transition probabilities

n-step transition probabilities

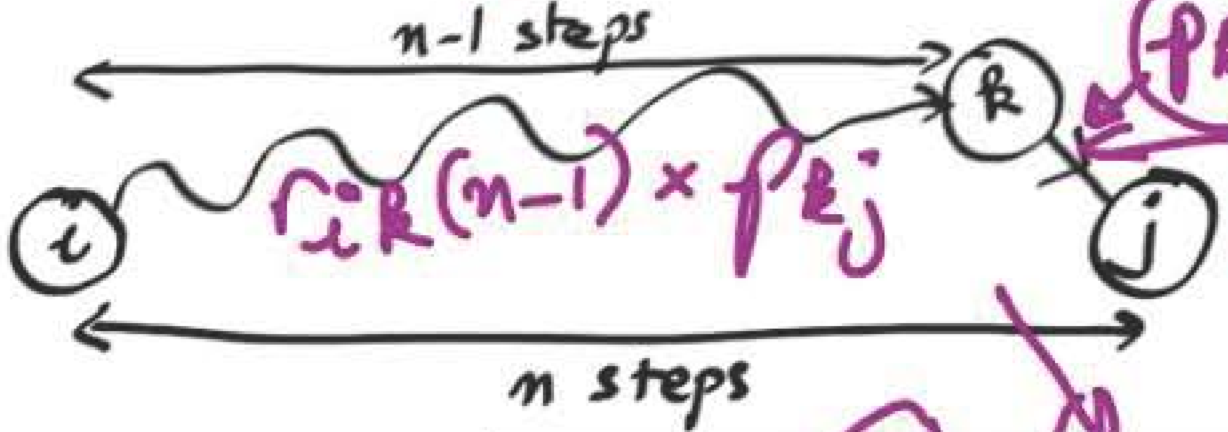
$r_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $\quad r_{ij}(1) = p_{ij} \ \forall i, \forall j$



- state probabilities, given initial state i:

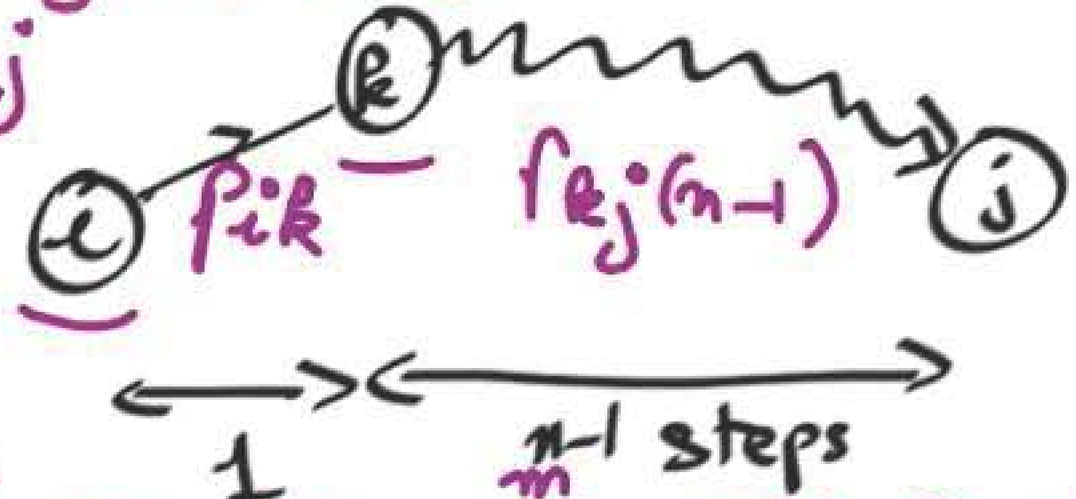
$$
\begin{aligned}
r_{ij}(n) &= \mathbf{P}(X_n = j \mid X_0 = i) \\
&= \mathbf{P}(X_{n+s} = j \mid X_s = i)
\end{aligned}
$$

$\sum_{j=1} r_{ij}(n) = 1 \quad \forall i, \forall n$



p_{kj}

$\mathbf{P}(X_n = j \mid X_0 = i, X_{n-1} = k)$ Markov $= \mathbf{P}(X_n = j \mid X_{n-1} = k) = p_{kj}$



$\left(r_{ij}(n) = \sum_{k=1}^{m} p_{ik}\, r_{kj}(n-1) \right)$

- key recursion: $\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}}$

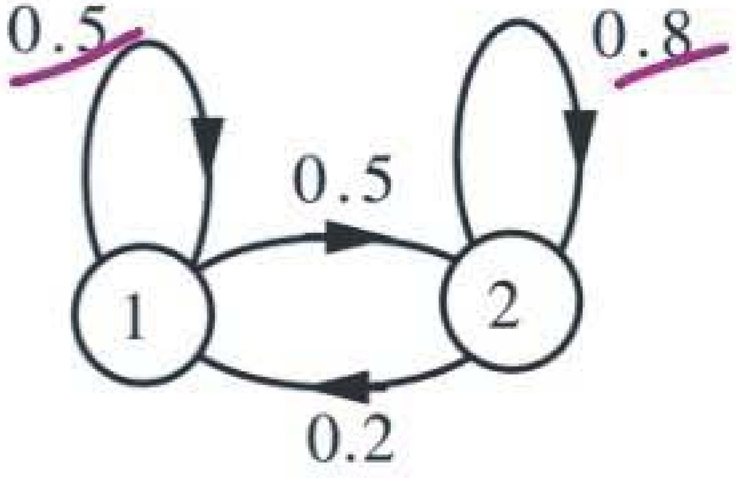
- random initial state:

initial distr

$$\mathbf{P}(X_n = j) = \sum_{i=1}^{m} \mathbf{P}(X_0 = i) r_{ij}(n)$$

example

$$\begin{cases} r_{11}(n) = r_{11}(n-1) \times 0.5 + r_{12}(n-1) \times 0.2 \\ r_{12}(n) = 1 - r_{11}(n) \end{cases}$$



$$r_{11}(101) = \frac{2}{7} \times 0.5 + \frac{5}{7} \times 0.2 = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

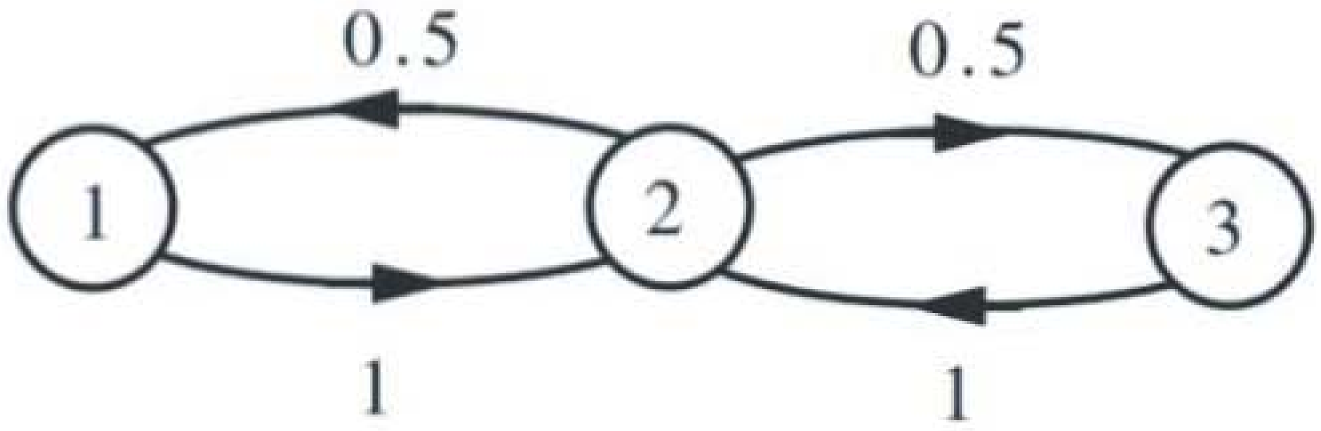
$r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$

	$n=0$	$n=1$	$n=2$	$n=100$	$n=101$
$r_{11}(n)$	1	0.5	$\xrightarrow{0.5}$ 0.25 $\xrightarrow{0.2}$ 0.10 0.35	$\approx 2/7$	? $2/7$
$r_{12}(n)$	0	0.5	0.65	$\approx 5/7$	? $5/7$
$r_{21}(n)$	0	0.2		$\approx 2/7$	
$r_{22}(n)$	1	0.8		$\approx 5/7$	

generic convergence questions

$r_{ij}(n) \xrightarrow[n\to\infty]{} \pi_j$?

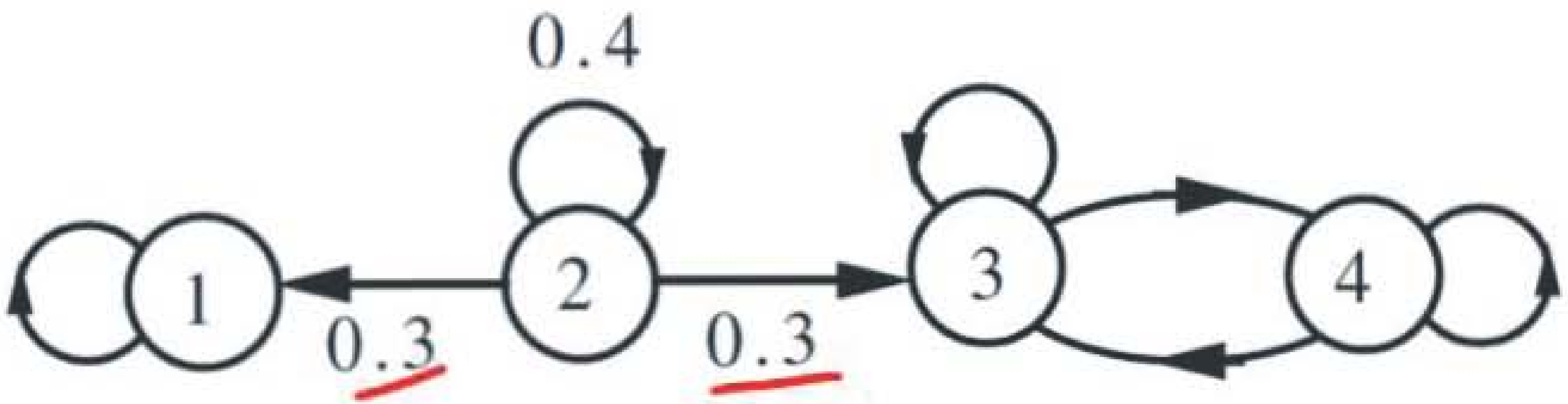
- does $r_{ij}(n)$ converge to something?



n odd: $r_{22}(n) = 0$

n even: $r_{22}(n) = 1$

- does the limit depend on initial state?



$r_{11}(n) = 1$

$r_{31}(n) = 0$

$r_{21}(n) = 1/2$

recurrent and transient states

- state i is recurrent if "starting from i, and from wherever you can go, there is a way of returning to i"
- if not recurrent, called transient